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PARTICLES AND ELECTROMAGNETIC FIELDS AROUND AXIAL-SYMMETRIC COMPACT GRAVITATING OBJECTS

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## INTRODUCTION

Topicality and demand of the theme of dissertation. Modern astronomical observations on myrovom level on the ground and space telescopes, and recent discoveries have provided convincing evidence that black holes have a significant impact on nearby objects around, emitting powerful gamma-ray bursts, absorbing the next star, and stimulating the growth of newborn stars in the surrounding areas. Study the photons motion around rotating black holes, in particular, the discovery and analysis of the form of silhouettes of these objects, setting and effective implementation of relevant radiostronomical observations on the proof of the existence of the black hole horizon and retrieval of information events on the central object in our galaxy within the Black Hole Cam (BHC) and Event Horizon Telesop (EHT) international projects is one of the most important tasks of modern astrophysics.

In the years of Independence, huge attention is paid to the development of theoretical physics and astronomy and basic research in these areas on a global level. In this regard has been achieved significant results in the field of relativistic astrophysics, in particular, developping a model of the magnetosphere of a neutron star, the analysis of the space-time structure and motion of test particles around black holes.

The study astrophysical processes in the vicinity of compact objects and their comparison with observational data is now one of the most important tasks in astrophysics of compact objects. The most important tasks is to conduct solid research work, in particular studies on following directions: to find the exact solutions describing the space-time around a gravitational compact objects; analysis of space-time structures in the framework of these decisions and find the equation of motion of test particles, such as photons; determining the silhouettes of black holes in general relativity and alternative theories of gravity; determination of the influence of the central object parameters and the plasma environment on the form of a silhouette; identify energy loss
dependence on the choice of the relativistic star of the gravity model. These objectives justify the topicality of the global level of scientific research.

This research work corresponds the tasks given by governmental regulatory documents, Orders of President of the Republic of Uzbekistan \# UP-559 "Onhigh level astronomical observatories and complexes of their service" from February 11, 1993, \# UP-4512 "On works further developing alternative sources of energies" from March 1, 2013.

Conformity of the research to the main priorities of science and technology development of the republic. The dissertation research has been carried out in accordance with the priority areas of science and technology in the Republic of Uzbekistan: II. "Power, energy and resource saving".

Review of international scientific researches on dissertation subject ${ }^{1}$. The electromagnetic field configurations in the external asymptotically uniform magnetic field, as well as shadows of rotating black holes has been carried out by the world's leading research centers and institutions of higher eduaction, in particular, Astronomical Institute, Albert Einstein Center and the Silesian University in Opava (Czech Republic), the University of Alberta (Canada), the Max Planck Institute for Gravitational Physics - Albert Einstein Institute and the Frankfurt University (Germany), Inter-University Centre of Astronomy and Astrophysics, Research Laboratory of Physics and Tata Institute of Fundamental Research (India), the Centre for Applied Space Technology and Microgravity and the Oldenburg University (Germany), State Astronomical Institute named after Sternberg of Moscow State

[^0]University (Russia), Institute of Nuclear Physics, Astronomical Institute and National University of Uzbekistan (Uzbekistan).

On the study of particles motion and electromagnetic fields around the black hole in external magnetic field it has been obtained a large number of original scientific results in the globe, including obtaining and study the properties of rotating black holes shadow in the framework of General Relativity and modified alternative theories of gravity (Max Planck Institute for Gravitational Physics - Albert Einstein Institute, Frankfurt University, Germany; Inter-University Centre for Astronomy and Astrophysics, Physics Research Laboratory, Tata Institute of Fundamental Research, India; the Centre for Applied Space Technology and Microgravity, Germany; State Astronomical Institute named after Sternberg of Moscow State University, Russia, AlFarabi Kazakh National University, Kazakhstan); it has been found the structure of the electromagnetic field around a rotating black hole and studied the equations of motion of charged particles around a rotating black hole in the presence of an external magnetic field (Astronomical Institute, Albert Einstein Center and the Silesian University in Opava, Czech Republic; University of Alberta, Canada; Inter-University Centre for Astronomy and Astrophysics, India; Centre for Applied Space Technology and Microgravity and Oldenburg University, Germany); study of energetic processes in the vicinity of rotating black holes in the framework of General Relativity and alternative theories of gravity, in particular, study of the properties of space-time properties around a black hole in the gravity model of Horava have been carried out (Albert Einstein Center and Silesian University in Opava, Czech Republic; University of Alberta, Canada; Max Planck Institute for Gravitational Physics - Albert Einstein Institute, Frankfurt University, Germany; Inter-University Centre for Astronomy and Astrophysics, Physics Research Laboratory, Tata Institute of Fundamental Research, India; the Centre for Applied Space Technology and Microgravity, Germany; State

Astronomical Institute named after Sternberg of Moscow State University, Russia, AlFarabi Kazakh National University, Kazakhstan).

Currently, in order to study shadows of rotating black holes, the particles motion and energetic processes in the vicinity of compact gravitational objects there have been carried out investigations in the world in a number of priority areas, including: the study of the photons motion and obtaining shadow of rotating black holes in the framework of general relativity and alternative theories of gravity and their analysis; theoretical modeling of electromagnetic fields around a compact gravitational objects and analysis of particle motion around these objects; study of energetic processes in the vicinity of rotating black holes in the presence of an external electromagnetic field; electromagnetic and gravitational radiation from compact gravitating objects.

Degree of study of the problem. Shadow of a rotating black hole with the various parameters of the central object, such as electric charge, brane charge, magnetic charge, in the framework of alternative theories of gravity have been studied by many scientists, for example from Japan (K. Hioki, K. Maeda), USA (J. Bardeen), Netherlands (H. Falcke), Germany (C. Laemmerzahl, V. Perlick, A. Grenzebach), Argentina (L. Amarilla, E. Eiroa), Italy (S. Bambi), Czech Republic (Z. Stuchlik, J. Schee), and others. However, all these works have been carried out in the framework of particular choice of solutions of compact objects and it does not exist a formalism describing shadows of black holes independent from the choice of the model of black holes, gravity theories and methods of measurement.

The solution of the electromagnetic field equations for rotating Kerr black hole in an external asymptotically uniform magnetic field, the study of charged particles motion around a rotating Kerr black hole immersed in an external uniform magnetic field, particles acceleration around a rotating black hole, the role of the magnetic field in the particles collision processes in the vicinity of slowly rotating black hole have
been studied by many scientists, for example, from the USA (M. Banados, J. Silk, S.M. West et al.), the UK (R. Wald et al.), Russia (D.V. Gal'tsov, Frolov, A. Aliyev), Turkey (N. Ozdemir), and others. However, a detailed study of the motion of charged particles and energy processes, such as the particles collision near a rotating black hole in an external magnetic field with a nonzero gravitomagnetic charge and/or deformation parameter has not been yet considered. The study of these effects would make constraints on the values of the various parameters of the black hole, such as gravitomagnetic charge and deformation parameter.

Investigations related to the study of the plasma effect on the photon motion around the compact objects in the framework of general relativity have been studied by various authors, in particular from Russia (G.S. Bisnovatyi-Kogan, O. Tsupko), Uzbekistan (B.J. Ahmedov, A.A. Tursunov, V.S. Morozov), Canada (A. Rogers), Germany (V. Perlick, J. Kunz), Dutch (H. Falcke), India (N. Dadhich, S. Ghosh, P. Joshi, M. Patil) and others. In these studies, however, there is no investigation related to the study of the impact of inhomogenous plasma on the optical properties of rotating black hole, in particular form a shadow of rotating black hole in the presence of an inhomogenous plasma.

The properties of space-time metric and the motion of the particles in the vicinity of a black hole within the Horava gravity model have been studied by several authors, for example from Czech Republic (Z. Stuchlik, J. Schee), Poland (M. Abramowicz et al.), Portugal (F. Lobo, T. Harko, F. Eiroa), Germany (S. Laemmerzahl, J. Kunz, E. Hackmann et al.), India (N. Dadhich, S. Ghosh, P. Joshi, M. Patil) and others. However, at this moment in the literature there is no research works addressed the study of the impact of the metric parameters of the gravity model on energetic processe and the role of the magnetic field in these processes.

Connection of the topic of dissertation with the scientific works of scientific research organizations, where the dissertation was carried out. The dissertation work was carried out in the framework of the scientific projects of the Institute of Nuclear Physics and Astronomical Institute: FA-F2-F079+F069 "Development of the equations of the electromagnetic and gravitational fields in relativistic astrophysics and cosmology, and phenomenological models of QCD in the description of hadrons and their interactions" (2007-2011); FA-F2-F058 "Study of Gravitational Lenses, Formed Galaxies and Generalized Gravita- tional Models" (2007-2011); FE2-FA-F134 "The particles motion and electromagnetic fields in the vicinity of relativis- tic stars and black holes in the alternative theories of gravity" (2012-2013); EF2-FA-0-12477 "Motion of particles with spin and propagation of electromagnetic waves in the vicinity of compact gravitational objects" (2014-2015); 1-10 "Particles and fields in the vicinity of relativistic gravitational objects from dark energy and wormholes" (2010-2011).

The aim of the research is the development of a theoretical formalism describing shadows of black holes and the identification of the physical laws of high-energetical processes in the vicinity of rotating black holes.

## The tasks of the research:

to develop a new coordinate-independent formalism to describe the shadow of a black hole, and define new parameters of distortion of black hole shadow;
to make a comparative analysis of the distortion parameters of a black hole shadow, proposed in the framework of the new formalism and parameters obtained by other authors;
to study the electromagnetic field and the charged particles motion in the vicinity of a deformed rotating black hole immersed in an external uniform magnetic field;
to consider the collision of electrically charged particles around a black hole with a non-zero gravitomagnetic charge immersed in an external magnetic field;
to determine the influence of an inhomogeneous plasma to the form of rotating black hole shadow;
to analyze the charged and neutral particles motion and collisions around Kehagias Sfetsos naked singularity in the presence of an external magnetic field;
to obtain estimates of the value of extracted energy from rotating black holes in the gravity model Horava;
to compare electromagnetic fields and spin down of the rotating relativistic compact stars.

The objects of the research are the black holes, neutron and strange relativistic stars.

The subjects of the research are the electromagnetic fields around compact objects, innermost stable circular orbits of the test particles around black holes, black hole shadow in the presence of plasma, energetic processes around rotating black holes in the presence of external magnetic field.

The methods of the research. On the theoretical level, the research methods are mathematical apparatus of macroscopic electrodynamics in general relativity and metric affine differential geometry, analytical and numerical methods for solving differential equations of motion and field.

The scientific novelty of the research is the follows:
For the first time a new coordinate-independent formalism to describe the shape of the black holes shadow has been developed and it was revealed that the first five coefficients of the polynomial expansion is sufficient to describe the properties of rotating black holes shadow with the accuracy of $\sim 0.1 \%$, it has been shown that the proposed definition of distortion of black holes shadow are stable under the signal noise;
it has been found that the observed size of the shadow of the black hole decreases due to the refraction of electromagnetic radiation in a plasma environment;
for the first time it was shown that for the high efficiency of the ultrahigh-energy processes relative to distant observers, both the non-existence of the horizon, and the strong rotational effects are necessary;
it was also shown that significant magnification of the efficiency of the ultra-high energy collisions is possible due to additional electromagnetic phenomena influencing collisions of charged particles;
for the first time it was shown that energy extraction through Penrose process is more realistic process among the energy extraction mechanisms from the rotating black hole in Horava-Lifshitz scenario; moreover, due to the Horava-Lifshitz gravity correction particles could be prevented from the infinite acceleration.
for the first time it was shown that the effect of compactness of strange star on the electromagnetic power loss of the star is non-negligible;
it was found that the strange star will lose more energy than typical rotating neutron star in general relativity.

Practical results of the research are as follows:
The analytical expressions for the vacuum electromagnetic fields of deformed rotating black holes in the external asymptotically uniform magnetic field hads been
obtained and it has been revealed that the induced electric field around the deformed black hole depends on the deformation parameter linearly, and the magnetic field squared;
expressions for energy and momentum, as well as radii of innermost stable circular orbits of charged particles in the vicinity of a black hole with gravitomagnetic charge immersed in external magnetic field has been obtained. It has been established that due to the existence of gravitomagnetic charge particle are prevented from acceleration to infinitely high energires;
it has been shown that the shape and size of the observed shadow of black hole varies depending on the plasma parameters, the rotation parameter of black hole and inclination angle between the observer plane and the axis of rotation of the black hole;
an upper limit for the deformation parameter of a rotating non-Kerr black hole has been obtained in the form $\varepsilon \leq 22$ using the comparison of the observation results on the radius of innermost stable circular orbits with the theoretical results;
it has been obtained the silhouettes of the rotating black holes shadow in the presence of an inhomogeneous plasma, which can be used to identify additional asymmetries in the shape of the shadow and retrieve information on the plasma parameters and the central compact object.

Reliability of the research results is provided by the followings: modern methods of general relativity and the theoretical physics and highly effective numerical methods and algorithms are used; careful check of a consistence of the received theoretical results with observational data and results of other authors is performed; conclusions are well consistent with the main provisions of the field theory of gravitational compact objects.

Scientific and practical significance of the research results. The scientific significance of the research results is determined by the ability of the developed formalism in the dissertation to analyze the black holes shadow obtained by a new generation of radiotelescopes in the millimeter diapason in the near future, and get an information on the various parameters and properties of the supermassive black holes at the center of our galaxy and galaxy M87. In addition, analysis of the silhouettes with the new formalism makes it possible to design new tests to verify the general relativity and other alternative theories of gravity.

The practical significance of the results of research lies in the fact that they can be used to obtain estimates of black holes of different parameters such as rotation, deformation, and gravitomagnetic charge, as well as the option that appears due to the higher-order corrections in Horava gravity model. Results can also be useful for the analysis of the nature and dynamics of the gravitational field, in the development of observational experiments and criteria for the detection and identification of strange stars.

Application of the research results. Ultra-high-energy collisions of particles in the field of near-extreme Kehagias-Sfetsos naked singularities and their appearance to distant observers have been studied in the frame of the program "Supporting Integration with the International Theoretical and Observational Research Network in Relativistic Astrophysics of Compact Objects" (2010-2014) (supported by the Operational Programme Education for Competitiveness funded by Structural Funds of the European Union and state budget of the Czech Republic and registered by number CZ.1.07/2.3.00/20.0071).

Shadows of the black holes and plasma influences have been used to obtain the trajectories of the photons around compact gravitating objectswithin the projects of the Central University of Delhi, India (Letter of Central University of Delhi, India from

September 20, 2016). This trajectories have been used to obtain the shadows of the black holes within the Born-Infield gravity theory.

The magnetosphere and slow down of the neutron stars have been used to develop the model of the neutron stars within the projets of Inter University Centre for Astronomy and Astrophysics (Letter from Inter University for Astronomy and Astrophysics, Pune, India from September 21, 2016). The models of the neutron stars have been used to obtain the value of the magnetic field at the stellar surface.

Approbation of the research results. The research results were reported in the form of reports and tested at 16 international and local scientific conferences, in particular: «Modern Problems of Physics and Astronomy» (Karshi, 2010), «Modern Problems of Modern Physics» (Samarkand, 2010), «Fusion and Plasma Physics» (Triest, 2011), 39-COSPAR general Assembly (Mysore, 2012), «Nuclear Science and Its Application» (Samarkand, 2012), International school on subnuclear physics (Erice, 2013), «General Relativity and Gravitation» (Warsaw, 2013), «Prague Synergy 2013: Accreting relativistic compact objects» (Prague, 2013), «Synergy Olomouc 2014» (Olomouc, 2014), 40-COSPAR General Assembly (Moscow, 2014), «International Congress of Mathematicians» (Seoul, 2014), "RAGTime - 2013" (Opava, 2013), "RAGTime - 2014" (Prague, 2014), "RAGTime - 2015" (Opava, 2015), "XII Marcel Grossman Meeting" (Rome, 2015), 41-COSPAR General Assembly (Istanbul, 2012).

The main results of the study were tested at the scientific seminars of the Institute of Nuclear Physics (2010-2016), Astronomical Institute (2010-2016), of the Department of Nuclear and Theoretical Physics of National University of Uzbekistan (2015-2016), Faculty of Philosophy and Science of Silesian University in Opava (Czech Republic, 2013-2016), Goethe University (Germany, 2013-2016), Max Planck Institute for Gravitational Physics (Germany, 2010-2016), Center for Applied Space

Technology and Microgravity (Germany, 2011), Tata Institute for Fundamental Research (India, 2014-2015), Inter-University Center for Astronomy and Astrophysics (India, 2010-2016), Delhi Central University (India, 2010-2016), International Centre for Theoretical Physics (Italy, 2011).

Publication of the research results. On the dissertation theme there were published 30 scientific works, including 15 scientific papers in international scientific journals recommended by the Supreme Attestation Commission of the Republic of Uzbekistan for publishing basic scientific results of doctoral theses.

Volume and structure of the dissertation. The dissertation consists of an introduction, four chapters, conclusion, two appendixes and a bibliography. The size of the dissertation is 181 pages.

List of published papers [1-15].

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10. Stuchlik Z., Schee J., Abdujabbarov A. Ultra-high-energy collisions of particles in the field of near-extreme Kehagias-Sfetsos naked singularities and their appearance to distant observers // Phys. Rev. D. - New York (USA), 2014. - vol. 89, N 10. - id.104048. - 19p.
11. Toshmatov B., Abdujabbarov A., Stuchlik Z., Ahmedov B. Quasinormal modes of test fields around regular black holes // Phys. Rev. D. - New York (USA), 2015. - vol. 91, N 8. - id.083008. - 13p.13. Toshmatov B., Abdujabbarov A., Ahmedov B., Stuchlik Z. Particle motion and Penrose processes around rotating regular black hole // Astrophysics and Space Science - Berlin Heidelberg: Springer (Germany), 2015. - vol. 357, N 1. - pp.1-15.
12. Abdujabbarov A., Rezzolla L., Ahmedov B. A coordinate-independent characterization of a black-hole shadow // Monthly Notices of the Royal Astronomical Society - London (UK), 2015. - vol. 454, N 3, pp. 2423-2435.
13. Atamurotov F., Ahmedov B., Abdujabbarov A. Optical properties of black holes in the presence of a plasma: The shadow // Physical Review D. - New York (USA), 2015. - vol. 92, N 8. - id. 084005. - 7p.

# CHAPTER I. PHOTON AND PARTICLE MOTION AROUND ROTATING BLACK HOLE 

### 1.1. Introduction

There is a widespread belief that the most convincing evidence for the existence of black holes will come from the direct observations of properties related to the horizon. These could be through the detection of gravitational waves from the collapse to a rotating star [16], from the ringdown in a binary black-hole merger [17], or through the direct observation of its "shadow". In a pioneering study, In [18] it was calculated the shape of a dark area of a Kerr black hole, that is, its "shadow" over a bright background appearing, for instance, in the image of a bright star behind the black hole. The shadow is a gravitationally lensed image of the event horizon and depends on the closed orbits of photons around the black hole ${ }^{2}$. Its outer boundary, which we will hereafter simply refer to as the shadow, corresponds to the apparent image of the photon capture sphere as seen by a distant observer. General relativity predicts, in fact, that photons circling the black hole slightly inside the boundary of the photon sphere will fall down into the event horizon, while photons circling just outside will escape to infinity. The shadow appears therefore as a rather sharp boundary between bright and dark regions and arises from a deficit of those photons that are captured by the event horizon. Because of this, the diameter of the shadow does not depend on the photons' energy, but uniquely on the angular momentum of the black hole. In general relativity and in an idealized setting in which everything is known about the emission properties of the plasma near the black hole, the shadow's diameter ranges from $4.5 r_{S}$ for an

[^1]extreme Kerr black hole, to $\sqrt{27} r_{S}$ for a Schwarzschild black hole, where $r_{S}:=2 G M / c^{2}$ is the Schwarzschild radius. In practice, however, the size and shape of the shadow will be influenced by the astrophysical properties of the matter near the horizon and, of course, by the theory of gravity governing the black hole.

Besides providing evidence on the existence of black holes, the observation of the black-hole shadow and of the deformations resulting in the case of nonzero spin, is also expected to help determine many of the black hole properties [see, e.g., [22, 23, 24]]. More specifically, imaging the shadow of a black hole via radio observations will allow one to test the predictions of general relativity for the radius of the shadow and study astrophysical phenomena in the vicinity of black holes [see [25] for a recent review]. In addition, it will allow one to set constraints on the validity of alternative theories of gravity which also predict black holes and corresponding shadows [see e.g. [26, 27, 23]].

The possible observation of a black-hole shadow has recently received a renewed attention as the spatial resolution attainable by very long baseline interferometry (VLBI) radio observations is soon going to be below the typical angular size of the event horizon of candidate supermassive black holes (SMBHs), such as the one at the center of the Galaxy or the one in the M87 galaxy [28]. These observations are the focus of international scientific collaborations, such as the Event Horizon Telescope (EHT) ${ }^{3}$ or the Black Hole Camera (BHC) ${ }^{4}$, which aim at VLBI observations at 1.3 mm and 0.87 mm of Sagittarius A* (Sgr A*) and M87. We recall that Sgr A* is a compact radio source at the center of the Galaxy and the SMBH candidate in our galaxy. In fact, the orbital motion of stars near Sgr A* indicates that its mass is $\cong 4.3 \cdot 10^{6} M_{\text {solar }}[29,30]$.

Given a distance of 8 kpc from us, the angular size of the Schwarzschild radius of the SMBH candidate in $\operatorname{Sgr} \mathrm{A}^{*}$ is $\sim 10 \mu$ as, so that the corresponding angular diameter

[^2]of the shadow is of the order of $\sim 50 \mu$ as. Similarly, with an estimated mass of $\cong 6.4$ $10^{9} M_{\text {solar }}$ [31] and a distance of 16 Mpc , the M87 galaxy represents an equally interesting SMBH candidate, with an angular size that is of the same order i.e., $\sim 40$ $\mu$ as. Although the resolution achievable at present is not sufficient to observe an image of the shadow of either black hole, it is sufficiently close that it is realistic to expect that near-future observations will reach the required resolution. Indeed, future EHT and BHC observations of Sgr A* are expected to go below the horizon scale and to start to provide precise information on the black-hole orientation, as well as on the astrophysical properties of the accretion flow taking place onto the black hole [32, 33].

An extensive literature has been developed to calculate the shadow of the black hole in known spacetimes, either within general relativity [34, 35, 36], or within alternative theories of gravity [37, 27, 38, 39, 40, 41, 42, 43]. In most cases, the expression of the shadow as a closed polar curve is not known analytically, but for the Plebanski-Demianski class of spacetimes, the shadow has been cast in an analytic form [44, 45].

Because the shadow is in general a complex polar curve on the celestial sky, an obvious problem that emerges is that of the characterization of its deformation. For example, in the case of a Kerr black hole, the difference in the photon capture radius between co-rotating and counter-rotating photons creates a "dent" on one side of the shadow, whose magnitude depends on the rotation rate of the black hole. A way to measure this deformation was first suggested in [46] and then further developed by other authors [47, 37]. In essence, in these approaches the shape of the shadow is approximated as a circle with radius $R_{s}$ and such that it crosses through three points located at the poles and at the equator of the shadow's boundary. The measure of the dent is then made in terms of the so called "deflection", that is, the difference between the endpoints of the circle and of the shadow, with a dimensionless distortion parameter being given by the ratio of the size of the dent to the radius $R_{s}$ [cf. Eq. (1.51)].

While this approach is reasonable and works well for a black hole such as the Kerr black hole, it is not obvious it will work equally well for black holes in more complex theories of gravity or even in arbitrary metric theories of gravity as those considered in [48]. Leaving aside the fact in all these works the shadow is assumed to be determined with infinite precision (an assumption which is obviously incompatible with a measured quantity), traditional approaches in characterizing the black-hole shadow and its deformations suffer from at least three potential difficulties: (i) they assume a primary shape, i.e., that the shadow can be approximated with a circle; (ii) they assume that the observer knows the exact position of the centre of black hole (an assumption that is unlikely to be true with real observational data); (iii) they are restricted to a very specific measure of the distortion and are unable to model arbitrary distortion.

To counter these potential difficulties, we present here a new general formalism that is constructed to avoid the limitations mentioned above. In particular, we assume that the shadow has an arbitrary shape and expand it in terms of Legendre polynomials in a coordinate system with origin in the effective centre of the shadow. This approach gives us the advantage of not requiring the knowledge of the center of the black hole and of allowing us to introduce a number of parameters that measure the distortions of the shadow. These distortions are both accurate and robust, and can be implemented in a coordinate independent manner by different teams analyzing the same noisy data.

A rotating astrophysical black hole without an electric charge is uniquely described by the Kerr metric, which only possesses two parameters, the total mass $M$ and the specific angular momentum a of black hole, within four-dimensional general relativity according to the no-hair theorem [49,50,51,52,53]. But in the regime of strong gravity, the general relativity could be broken down and astrophysical black holes might not be the Kerr black holes as predicted by the no-hair theorem [54, 55, 56].

Recently, Johannsen and Psaltis proposed a deformed Kerr-like metric suitable for the strong field of the no-hair theorem, which describes so called rotating non-Kerr black hole [55]. The study of the particle orbits could provide an opportunity for constraining the allowed parameter space of solutions, and to provide a deeper insight into the physical nature and properties of the corresponding spacetime metrics. Therefore, in this work we paid attention for studying the electromagnetic field and charged particle motion around rotating non-Kerr black hole immersed in external magnetic field. In the recent paper [57] the properties of the ergosphere and energy extraction by the Penrose process in a rotating non-Kerr black hole have been investigated. Direct imaging rapidly-rotating non-Kerr black holes and their shadows are studied in the paper [58]. Strong gravitational lensing by a rotating non-Kerr compact object are investigated in [59]. The strong dependence of the predicted energy spectra and energy-dependent polarization degree and polarization direction on the parameters of rotating non-Kerr black hole is found in [60]. The brief review on testing the Kerr black hole hypothesis is given in [61]. The accretion disc properties around rotating non-Kerr compact object are considered by the authors of [62].

In principle, the properties of innermost stable circular orbits (ISCO) could provide a good tool for understanding the energetic processes of black hole. The acceleration of particles, circular geodesics, accretion disk, and high-energy collisions in the Janis-Newman-Winicour spacetime have been considered in [63, 64]. Study of the motion of the test particles and particle acceleration mechanisms in axial-symmetric spacetime may provide new tool for studying new important general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian analogues [65, 66]. It has been recently shown in [67] that primordial Kerr superspinars, extremely compact objects with exterior described by the Kerr nakedsingularity geometry, can serve as efficient accelerators for extremely high-energy collisions. The properties of the event horizon, static limit for a charged rotating black
hole solution of minimal supergravity theory and particle motion as well have been considered in [68].

At present there is no any observational evidence for the existence of gravitomagnetic monopole, i.e. so-called NUT [69] parameter or magnetic mass. Therefore study of the motion of the test particles and particle acceleration mechanisms in NUT spacetime may provide new tool for studying new important general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian analogues (see, e.g. [65, 66, 70, 71, 72, 73, 74] where solutions for electromagnetic waves and interferometry in spacetime with NUT parameter have been studied.). Kerr-Taub-NUT spacetime with Maxwell and dilation fields is recently investigated by authors of the paper [75]. In papers [76, 77] the plasma magnetosphere around a rotating, magnetized neutron star and charged particle motion around compact objects immersed in external magnetic field have benn studied in the presence of the NUT parameter. The Penrose process in the spacetime of rotating black hole with nonvanishing gravitomagnetic charge has been considered in [78]. The electromagnetic field of the relativistic star with nonvanishing gravitomagnetic charge has been considered by [79].

Astrophysical processes which may produce high energy radiation near a rotating black hole horizon attract more attention in recent publications. The processes which are related to the effect of Penrose [80] have been properly considered in [81, 82, 83]. Recently Banados, Silk and West (BSW) [84] pointed out that the collisions of particles near extremely rotating black hole can produce particles of high center-of-mass energy. The results of [84] have been commented in [85] where authors concluded that astrophysical limitations on the maximal spin, back-reaction effects and sensitivity to the initial conditions impose severe limits on the likelihood of such accelerations.

This chapter is devoted to examine an above mentioned effect of particle acceleration in the presence of gravitomagnetic charge for the case when the collision
of particles occurs in the vicinity of nonrotating black hole embedded in magnetic field. It is very interesting to study the electromagnetic fields and particle motion in NUT space with the aim to get new tool for studying new important general relativistic effects. Moreover this demonstration can be interesting because of the existence of both theoretical and experimental evidences that a magnetic field must be present in the vicinity of black holes. Note, that hereafter we use the weak magnetic field approximation in such sense that the energy-momentum of this field does not change the background geometry of black hole. For a black hole with mass M this condition means that the strength of magnetic field satisfies to the following condition [86, 87]

$$
\begin{equation*}
B \ll B_{\max }=\frac{c^{4}}{G^{3 / 2} M_{\odot}}\left(\frac{M_{\odot}}{M}\right) \sim 10^{19} \frac{M_{\odot}}{M} \text { Gauss . } \tag{1.1}
\end{equation*}
$$

Following to [87], we call these black holes as weakly magnetized. One can say that this condition is quite general and satisfies for both stellar mass and supermassive black holes.

The chapter is based on the following papers $[1,6,7,14]$ of the author and organized as follows. In Sect. 1.2 we develop the new coordinate-independent formalism, where an arbitrary black-hole shadow is expanded in terms of Legendre polynomials. Using this formalism, we introduce in Sect. 1.3 various distortion parameters of the shadow. In Sect. 1.4 we apply the formalism to a number of blackhole spacetimes by computing the coefficients of the expansion and by showing that they exhibit an exponentially rapid convergence. We also compare the properties of the different distortion parameters and assess which definition appears to be more accurate and robust in general. Section 1.5 offers a comparison between the new distortion parameters introduced here with the more traditional ones simulating the noisy data that are expected from the observations. The equations of motion of charged particles and
their motion at the equatorial plane in the vicinity of the rotating non-Kerr black hole have been considered in Sect. 1.6. We obtain the effective potential for charged test particle with a specific angular momentum, orbiting around the black hole, as a function of the external magnetic field, deformation parameter, and angular momentum of nonKerr black hole. In Sect. 1.7 we find the exact expression for dependence of minimal radius of circular orbit from the parameters of spacetime metric around rotating nonKerr black hole. Sect. 1.8 is devoted to study of electromagnetic field and charged particle motion in the magnetized black holes with NUT parameter, with the main focus on the properties of their ISCOs. Then we considered particles collisions in the vicinity of a weakly magnetized black hole with nonvanishing gravitomagnetic charge. Finally, Sect. 1.9 summarizes our main results and the prospects for the use of the new formalism.

### 1.2. General characterization of the shadow

In what follows we develop a rather general formalism to describe the blackhole shadow that radio astronomical observations are expected to construct. For all practical purposes, however, we will consider the problem not to consist of the determination of the innermost unstable circular orbits for photons near a black hole. Rather, we will consider the problem of characterizing in a mathematically sound and coordinateindependent way a closed curve in a flat space, as the one in which the image will be available to us as distant observers.

Assume therefore that the astronomical observations provide the shadow as an one-dimensional closed curve defined by the equation

$$
\begin{equation*}
R^{\prime}=R^{\prime}\left(\psi^{\prime}\right), \tag{1.2}
\end{equation*}
$$

where $R^{\prime}$ and $\psi^{\prime}$ can be thought of as the radial and angular coordinates in a polar coordinate system with origin in O0. In practice, astronomical observations will not be able to provide such a sharp closed line and a more detailed analysis would need to take the observational uncertainties (which could well be a function of 0 ) into account. We will discuss some of these uncertainties in Sect. 1.5, but for the time being we will consider the shadow as an idealized one-dimensional curve. A schematic example of the polar curve is shown in Fig. 1.1, where $\alpha^{\prime}$ and $\beta^{\prime}$ are the so-called "celestial coordinates" of the observer, and represent an orthogonal coordinate system with one of the unit vectors being along the line of sight.

Of course, there is no reason to believe that such a coordinate system is particularly useful, or that in using it a nonrotating black hole will have a shadow given by a perfect circle. Hence, in order to find a better coordinate system, and, in particular, one in which a Schwarzschild black hole has a circular shadow, we define the effective center of the curve in strict analogy with the definition of the center of mass in a collection of point particles. More specifically, if the closed curve is composed of $N$ intervals, each with length $\Delta l_{i}$

$$
\begin{equation*}
\Delta l_{i}:=\int_{\psi_{i, 1}}^{\psi_{i, 2}} \sqrt{g_{\psi^{\prime}} \psi^{\prime}} d \psi^{\prime}, \tag{1.3}
\end{equation*}
$$

where $g_{\psi^{\prime} \psi^{\prime}}$ is the polar metric function, then the position of the effective center is simply given by

$$
\begin{equation*}
\overrightarrow{\boldsymbol{R}}_{0}:=\frac{\sum_{i=1}^{N} \overrightarrow{\boldsymbol{R}}_{i}^{\prime} \Delta l_{i}}{\sum_{i=1}^{N} \Delta l_{i}}, \tag{1.4}
\end{equation*}
$$

where $\overrightarrow{\boldsymbol{R}}_{\boldsymbol{i}}{ }_{i}$ is the radial vector in the $O^{\prime}$ coordinate system selecting the $i$-th element of the curve. The extension of expression (1.3) to a continuum is then straightforward and
once again mimics the corresponding extension of the center of mass for a solid body, i.e.,

$$
\begin{equation*}
\overrightarrow{\boldsymbol{R}}_{0}:=\frac{\int_{0}^{2 \pi} \overrightarrow{\boldsymbol{e}}_{R^{\prime}} R^{\prime 2} d \psi^{\prime}}{\int_{0}^{2 \pi} R^{\prime} d \psi^{\prime}} \tag{1.5}
\end{equation*}
$$

where $\vec{e}_{R^{\prime}}$ is the radial-coordinate unit vector.

Figure 1.1
Schematic representation of the black-hole shadow as a generic polar curve $R_{\psi}$ in a coordinate system with origin $O$ in the "center" of the shadow. The latter is translated by a vector $\overrightarrow{\boldsymbol{R}}_{0}$ with respect to the arbitrary coordinate system with origin $O^{\prime}$ in which the observations are made.


From the knowledge of the vector $\overrightarrow{\boldsymbol{R}}_{\mathbf{0}}$, the coordinate position of the effective center can be expressed explicitly in terms of the radial and angular coordinates as

$$
\begin{align*}
R_{0}:= & \left(\int_{0}^{2 \pi} R^{\prime} d \psi^{\prime}\right)^{-1}\left[\left(\int_{0}^{2 \pi} R^{\prime 2} \cos \psi^{\prime} d \psi^{\prime}\right)^{2}+\left(\int_{0}^{2 \pi} R^{\prime 2} \sin \psi^{\prime} d \psi^{\prime}\right)^{2}\right]^{1 / 2}  \tag{1.6}\\
& \psi_{0}:=\tan ^{-1}\left(\frac{\int_{0}^{2 \pi} R^{\prime 2} \sin \psi^{\prime} d \psi^{\prime}}{\int_{0}^{2 \pi} R^{\prime 2} \cos \psi^{\prime} d \psi^{\prime}}\right) \tag{1.7}
\end{align*}
$$

We note that if the center of the primary coordinate system $O^{\prime}$ coincides with the black-hole origin, then the parameter $R_{0}$ exactly corresponds to the shift of the center of the shadow with respect to the black-hole position defined in [27].

Having determined the effective center of the shadow, it is convenient to define a new polar coordinate system centered in it with coordinates $(R, \psi)$. Clearly, the new coordinate system with origin $O$ is just translated by $R_{0}$ with respect to $O^{\prime}$ and, hence, the relation between the two coordinate systems is given by

$$
\begin{align*}
R & :=\left[\left(R^{\prime} \cos \psi^{\prime}-R_{0} \cos \psi_{0}\right)^{2}+\left(R^{\prime} \sin \psi^{\prime}-R_{0} \sin \psi_{0}\right)^{2}\right]^{1 / 2}  \tag{1.8}\\
\psi & :=\tan ^{-1} \frac{R^{\prime} \sin \psi^{\prime}-R_{0} \sin \psi_{0}}{R^{\prime} \cos \psi^{\prime}-R_{0} \cos \psi_{0}} \tag{1.9}
\end{align*}
$$

Note that we have kept the new axes and parallel to the original ones $\alpha^{\prime}$ and $\beta^{\prime}$. This is not strictly necessary but given the arbitrarity of the orientation of both sets of axes, it provides a useful simplification.

A well-defined center of coordinates allows us now to obtain a robust definition of the reference areal circle as the circle having the same area as the one enclosed by the shadow. In particular, given the closed parametric curve $R=R(\psi)$, its area will be given by

$$
\begin{equation*}
\mathcal{A}:=\frac{1}{2} \int_{\psi_{1}}^{\psi_{2}} R^{2} d \psi=\frac{1}{2} \int_{\lambda_{1}}^{\lambda_{2}} R^{2}(\lambda)\left(\frac{d \psi}{d \lambda}\right) d \lambda, \tag{1.10}
\end{equation*}
$$

where the second equality considers the representation of the curve in terms of a more generic parameter $\lambda$, i.e., $R=R(\psi(\lambda))$, and where the integration limits $\lambda_{1,2}$ can be found from the condition $\psi(\lambda)=0$ and $\psi(\lambda)=2 \pi$, respectively. We can then define the areal radius $R_{A}$ of the reference circle simply as

$$
\begin{equation*}
R_{\mathcal{A}}:=\left(\frac{A}{\pi}\right)^{1 / 2} \tag{1.11}
\end{equation*}
$$

Similarly, and if simpler to compute, it is possible to define the circumferential radius $R_{c}$ of the reference circle as

$$
\begin{equation*}
R_{\mathcal{C}}:=\frac{\mathcal{C}}{2 \pi} \tag{1.12}
\end{equation*}
$$

where the circumference is calculated as

$$
\begin{equation*}
\mathcal{C}:=\int\left(g_{R R} d R^{2}+g_{\psi \psi} d \psi^{2}\right)^{1 / 2}=\int_{\lambda_{1}}^{\lambda_{2}}\left[\left(\frac{d R}{d \lambda}\right)^{2}+R^{2}\left(\frac{d \psi}{d \lambda}\right)^{2}\right]^{1 / 2} d \lambda . \tag{1.13}
\end{equation*}
$$

An areal radius is particularly useful as it enables one to measure two useful quantities, namely, the local deviation of the shadow $R:=R(\psi)$ from the areal circle, i.e.,

$$
\begin{equation*}
D_{\psi}=D(\psi):=\left|R_{\mathcal{A}}-R_{\psi}\right|, \tag{1.14}
\end{equation*}
$$

and its polar average

$$
\begin{equation*}
D_{\langle\psi\rangle}:=\frac{1}{2 \pi} \int_{0}^{2 \pi} D_{\psi} d \psi=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|R_{\mathcal{A}}-R_{\psi}\right| d \psi . \tag{1.15}
\end{equation*}
$$

Note that although similar, the areal and the circumferential radii are in general different and coincide just for a spherically symmetric black hole, in which case $R_{A}=R_{C}=R_{\psi}$, and of course $D_{\psi}=0=D_{s}$. All of these geometrical quantities are shown schematically in Fig. 1.2.

### 1.3. Distortion parameters

With a well defined and unambiguous set of coordinates $(R, \psi)$ we can next move to the characterization of the geometrical properties of the shadow. To this scope we simply employ an expansion in terms of Legendre polynomial, i.e., we define

$$
\begin{equation*}
R_{\psi}:=\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(\cos \psi), \tag{1.16}
\end{equation*}
$$

where $P_{\ell}(\cos \psi)$ is the Legendre polynomial of order $\ell$ and the coefficients $c_{\ell}$ of the expansion (1.16) can be found as
$c_{\ell}:=\frac{2 \ell+1}{2} \int_{0}^{\pi} R(\psi) P_{\ell}(\cos \psi) \sin \psi d \psi=\frac{2 \ell+1}{2} \int_{\lambda_{1}}^{\lambda_{2}} R(\lambda) P_{\ell}(\cos \psi) \sin \psi\left(\frac{d \psi}{d \lambda}\right) d \lambda$.

The integration limits $\lambda_{1,2}$ can be found from the condition $\psi(\lambda)=0$ and $\psi(\lambda)=\pi$, respectively. Using this decomposition, it is straightforward to measure the differences
between the value of the parametrized shadow at two different angles. For example, the relative difference between the shadow at $\psi=0$ and at a generic angle $\psi=\pi / m$ can be computed simply as

$$
\begin{equation*}
\delta_{m}:=\frac{R_{\psi}(\psi=0)-R_{\psi}(\psi=\pi / m)}{R_{\psi}(\psi=0)}=1-\frac{\left.\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(\cos \psi)\right|_{\psi=\pi / m}}{\left.\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(\cos \psi)\right|_{\psi=0}} . \tag{1.18}
\end{equation*}
$$

Figure 1.2
Schematic representation of the local distortion $D_{\psi}$ between the polar curve $R_{\psi}$ representing the black-hole shadow and representative circles with circumference and area radii $R_{C}$ and $R_{A}$, respectively


When $m=1$, expression (1.18) simplifies to

$$
\begin{equation*}
\delta_{1}:=1-\frac{\sum_{\ell=0}^{\infty}(-1)^{\ell} c_{\ell}}{\sum_{\ell=0}^{\infty} c_{\ell}}, \tag{1.19}
\end{equation*}
$$

while, when $m=2$, the difference can still be computed analytically and is given by

$$
\begin{equation*}
\delta_{2}:=1-\frac{\mathscr{B}}{\mathscr{A}}, \tag{1.20}
\end{equation*}
$$

where we have introduced the following and more compact notation that will be used extensively in the remainder

$$
\begin{align*}
& \mathscr{A}:=R_{\psi}(\psi=0)=\sum_{\ell=0}^{\infty} c_{\ell},  \tag{1.21}\\
& \mathscr{B}:=R_{\psi}(\psi=\pi / 2)=\sum_{\ell=0}^{\infty}(-1)^{\ell} \frac{(2 \ell)!}{2^{2 \ell}(\ell!)^{2}} c_{2 \ell},  \tag{1.22}\\
& \mathscr{C}:=R_{\psi}(\psi=3 \pi / 2)=\sum_{\ell=0}^{\infty}(-1)^{\ell} c_{\ell} . \tag{1.23}
\end{align*}
$$

Analytic expressions for (1.18) when $m>2$ are much harder to derive, but can be easily computed numerically.

We note that the parametrization (1.18) is quite general and allows us to recover in a single definition some of the expressions characterizing the distortion of the shadow and that have been introduced by other authors. For example, the parameter $\delta_{n}^{1}$ can be associated to the distortion parameter $\delta$ first introduced in [46] [cf. Fig. 3 of [46]]. Similarly, the parameter $\delta_{4}$ is directly related to the distortion parameter $\varepsilon$ introduced in [27] [cf. Fig. 3 of [27]].

In what follows we will exploit the general expression for the polar curve representing the black-hole shadow to suggest three different definitions that measure in a coordinate-independent manner the amount of distortion of the shadow relative to some simple background curve, e.g., a circle. These expressions are all mathematically
equivalent and the use of one over the other will depend on the specific properties of the observed shadow.

### 1.3.1 Distortion parameter - I

We start by considering three points on the polar curve $\mathrm{A}, \mathrm{B}$, and D , which occupy precise angular positions at $\psi=0 ; \pi / 2$, and $3 \pi / 2$, respectively (see diagram in Fig. 1.3). The corresponding distances $\mathrm{OA}, \mathrm{OB}$ and OD from the center of coordinates O can then be expressed as

$$
\begin{align*}
& R_{A}:=R_{\psi}(\psi=0)=\left.\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(\cos \psi)\right|_{\psi=0}=\mathscr{A},  \tag{1.24}\\
& R_{B}:=R_{\psi}(\psi=\pi / 2)=\left.\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(\cos \psi)\right|_{\psi=\pi / 2}=\mathscr{B},  \tag{1.25}\\
& R_{D}:=R_{\psi}(\psi=3 \pi / 2)=R_{B}, \tag{1.26}
\end{align*}
$$

where in the last equality we have exploited the fact that the expansion in Legendre polynomials is symmetric with respect to the $\alpha$ axis.

Next, we define a new parametric curve for which $R_{A}=R_{B}=R_{D}$ and thus that satisfies the following condition

$$
\begin{equation*}
\mathscr{B}=\mathscr{A}, \tag{1.27}
\end{equation*}
$$

or, equivalently, for $\ell>0$

$$
\begin{equation*}
c_{2 \ell-1}=c_{2 \ell}\left[(-1)^{\ell} \frac{(2 \ell)!}{2^{2 \ell}(\ell!)^{2}}-1\right] . \tag{1.28}
\end{equation*}
$$

Figure 1.3
Schematic representation of one of the distortion parameters. The quantity $D_{s, I}$ measures the difference between the Legendre expanded polar curve $R_{\psi, I}$ and the reference circle of radius $R_{s, I}$ and passing through the points $\mathrm{A}, \mathrm{B}$, and D .


The corresponding polar expression, formulated in terms of the Legendre polynomials expansion (1.16), is therefore given by

$$
\begin{equation*}
R_{\psi, I}(\psi)=c_{0}+\sum_{\ell=1}^{\infty} c_{2 \ell-1}\left[P_{2 \ell-1}(\cos \psi)+\left((-1)^{\ell} \frac{(2 \ell)!}{2^{2 \ell}(\ell!)^{2}}-1\right)^{-1} P_{2 \ell}(\cos \psi)\right] . \tag{1.29}
\end{equation*}
$$

To measure the distortion we need a reference curve, which we can choose to be the circle passing through the three points $\mathrm{A}, \mathrm{B}$, and D and thus with radius

$$
\begin{equation*}
R_{S, 1}:=R_{A}=\mathscr{B}=R_{B}=\mathscr{A} . \tag{1.30}
\end{equation*}
$$

We can now compute the deviation of the parametric curve (1.29) from the corresponding background circle of radius $R_{s, I}$ at any angular position. However, as customary in this type of considerations, we can consider the shadow to be produced by a rotating black hole with spin axis along the $\beta$ axis, so that the largest deviations will be on the axis of negative $\alpha$ (see Fig. 1.3). More specifically, we can define the difference between the curves at $\psi=\pi$ as

$$
\begin{equation*}
D_{s, 1}:=R_{s, \mathrm{I}}-R_{\psi, I}(\psi=\pi)=\mathscr{B}-\sum_{\ell=0}^{\infty} c_{2 \ell}+\sum_{\ell=1}^{\infty} c_{2 \ell-1}=2 \sum_{\ell=1}^{\infty} c_{2 \ell-1} . \tag{1.31}
\end{equation*}
$$

It follows that our first definition for the dimensionless distortion parameter $\delta_{s, I}$ can then be given by

$$
\begin{equation*}
\delta_{s, \mathrm{I}}:=\frac{D_{s, 1}}{R_{s, 1}}=\frac{2 \sum_{\ell=1}^{\infty} c_{2 \ell-1}}{\mathscr{B}}=: \sum_{\ell=1}^{\infty} \delta_{\ell, I}, \tag{1.32}
\end{equation*}
$$

which reduces to the compact expression

$$
\begin{equation*}
\delta_{s_{, 1}} \simeq \frac{2 c_{1}}{c_{0}}=\delta_{1, \mathrm{I}} . \tag{1.33}
\end{equation*}
$$

when only the first two coefficients in the expansion are taken into account, i.e., for $c_{0} \neq 0 \neq c_{1}$ and $c_{\ell}=0$, with $\ell \geq 2$ (note that $\delta_{0, I}=0$ ).

### 1.3.2 Distortion parameter - II

A second possible definition of the distortion parameter is slightly more general and assumes that the radial distance of points A and B from the center of coordinates is not necessarily the same, i.e., $R_{A} \neq R_{B}$. In this case, one can think of introducing a new point E on the axis (not shown in Fig. 1.3), such that the distances $\mathrm{AE}=\mathrm{EB}$ and which could therefore serve as the center of the reference circle. Since the values of the coordinates RA and RB are defined by expressions (1.24) and (1.25), we can use the condition $\mathrm{AE}=\mathrm{EB}$ to find that one can easily find position of the point E on the $\alpha$ axis is given by

$$
\begin{equation*}
R_{E}=\left|\frac{R_{B}^{2}-R_{A}^{2}}{2 R_{A}}\right|, \tag{1.34}
\end{equation*}
$$

with the corresponding angular position $\psi_{E}$ being either 0 or $\pi$, i.e.,

$$
\begin{equation*}
\psi_{E}=\cos ^{-1}\left(\frac{R_{A}-R_{B}}{\left|R_{A}-R_{B}\right|}\right) . \tag{1.35}
\end{equation*}
$$

The radius of the circle passing through the three points $\mathrm{A}, \mathrm{B}$, and D is

$$
\begin{equation*}
R_{s, \text { II }}=\frac{R_{B}^{2}+R_{A}^{2}}{2 R_{A}}, \tag{1.36}
\end{equation*}
$$

the shadow deviation at $\psi=\pi$ from the circle of radius $R_{s, I I}$ can be found using relation

$$
\begin{equation*}
D_{s, \text { II }}=2 R_{s, \text { III }}-\left(R_{A}+R_{C}\right) . \tag{1.37}
\end{equation*}
$$

Finally, we can introduce the distortion parameter $D_{s, I I}$ defined as

$$
\begin{equation*}
D_{s, 11}:=\frac{\mathscr{B}^{2}}{\mathscr{A}}-\mathscr{C}, \tag{1.38}
\end{equation*}
$$

so that the second dimensionless distortion parameter is expressed as

$$
\begin{equation*}
\delta_{s, \text { II }}:=\frac{D_{s, \text { II }}}{R_{s, \text { II }}}=2\left(\frac{\mathscr{B}^{2}-\mathscr{A} \mathscr{C}}{\mathscr{B}^{2}+\mathscr{A}^{2}}\right) . \tag{1.39}
\end{equation*}
$$

The expression for the dimensionless distortion (1.39) is in this case more complex that the one presented in (1.32); however, in the simpler case in which only the lowest order coefficients are retained, i.e., if $c_{0} \neq 0 \neq c_{1}$ and $c_{\ell}=0$, with $\ell \geq 2$, we have

$$
\delta_{s, \text { II }} \simeq \frac{2 c_{1}^{2}}{2 c_{0}^{2}+2 c_{0} c_{1}+c_{1}^{2}}=\delta_{1, \text { II }} .
$$

### 1.3.3 Distortion parameter - III

A third and final possible definition of the distortion parameter is one that is meant to consider the case in which the shadow is still reflection symmetric relative to the $\alpha$ axis, but does not cross the $\beta$ axis with a zero slope. Rather, the curve admits a point, say S , at angular position $0<\psi_{S}<\pi$, where it has zero slope relative to the $(\alpha, \beta)$ coordinate system. This point will be referred to as the "slope point" of the parametric curve $R_{\psi}$ representing the shadow.

To compute the position of this point in the $(\alpha, \beta)$ coordinates we simply need to find the solution to the equation

$$
\begin{equation*}
\left.\frac{d \beta}{d \alpha}\right|_{\psi_{S}}=0 \tag{1.41}
\end{equation*}
$$

or, equivalently, solve for the differential equation

$$
\begin{equation*}
\frac{d R_{\psi}}{d \psi} \sin \psi+R_{\psi} \cos \psi=0 \tag{1.42}
\end{equation*}
$$

Using the expansion in terms of Legendre polynomials (1.16), we can rewrite (1.42) as

$$
\begin{equation*}
\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}(x) x-\sum_{\ell=0}^{\infty} c_{\ell} \frac{d P_{\ell}(x)}{d x}\left(1-x^{2}\right)=0 \tag{1.43}
\end{equation*}
$$

where we have set $\mathrm{x}:=\cos$. The solutions of (1.43) provide the positions of all the possible slope points in the parametric curve, and the solution is unique in the case in which the shadow $R()$ is convex. The corresponding coordinates of the point $S$ are then

$$
\begin{align*}
R_{S} & =\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right),  \tag{1.44}\\
\psi_{A} & =\cos ^{-1}\left(x_{S}\right) . \tag{1.45}
\end{align*}
$$

As for the second distortion parameter in Sect. 1.3.3, we set E to be the center of the circle passing through the points $\mathrm{A}, \mathrm{S}$, and $S^{\prime}$, where $S^{\prime}$ the point is symmetric to the point $S$ with respect to the $\alpha$ axis. Using the condition $\mathrm{AE}=\mathrm{ED}$, we obtain the solution

$$
\begin{align*}
R_{D} & =\left|\frac{\mathscr{A}^{2}-\left(\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)\right)^{2}}{2 \sum_{\ell=0}^{\infty} c_{\ell}\left(1-P_{\ell}\left(x_{S}\right) x_{S}\right)}\right|,  \tag{1.46}\\
\psi_{D} & =\cos ^{-1}\left(\frac{R_{D}}{\left|R_{D}\right|}\right) . \tag{1.47}
\end{align*}
$$

Also for this third case, the radius of the circle $R_{s, I I I}$ passing through the three points A, S , and $S^{\prime}$, the distortion parameters $D_{s, I I I}$, and $\delta_{\mathrm{s}, I I I}$, have respectively the form

$$
\begin{gather*}
R_{s, \text { III }}=\frac{\mathscr{A}^{2}-2 x_{S} \mathscr{A}\left(\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)\right)+\left(\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)\right)^{2}}{2 \sum_{\ell=0}^{\infty} c_{\ell}\left(1-P_{\ell}\left(x_{S}\right) x_{S}\right)},  \tag{1.48}\\
D_{s, \text { III }}=2 R_{s, \text { III }}-\left(R_{A}+R_{C}\right)=\left(\sum_{0=1}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)\right) \frac{\left(\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)-x_{S} \sum_{\ell=1}^{\infty} c_{2 \ell-1}-\mathscr{A} \mathscr{C}\right)}{\sum_{\ell=0}^{\infty} c_{\ell}\left(1-P_{\ell}\left(x_{S}\right) x_{S}\right)},  \tag{1.49}\\
\delta_{s, \text { III }}=\frac{D_{s, \text { III }}}{R_{s, \text { III }}}=2\left(\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)\right) \frac{\left(\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)-x_{S} \sum_{\ell=1}^{\infty} c_{2 \ell-1}-\mathscr{A} \mathscr{A}\right)}{\mathscr{A}^{2}-2 x_{S} \mathscr{A} \sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)+\left(\sum_{\ell=0}^{\infty} c_{\ell} P_{\ell}\left(x_{S}\right)\right)^{2}} . \tag{1.50}
\end{gather*}
$$

We note that this definition is similar to the one proposed in [46], who measure the dimensionless distortion of the shadow as

$$
\begin{equation*}
\delta_{s, \mathrm{HM}}:=\frac{D_{s, \mathrm{HM}}}{R_{s, \mathrm{HM}}}, \tag{1.51}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{s, \mathrm{HM}}:=R_{\psi}(\psi=\pi)-R_{s, \mathrm{HM}}, \tag{1.52}
\end{equation*}
$$

and with $R_{s, H M}$ being the radius of the circle passing through the points $\mathrm{A}, \mathrm{S}$, and $S^{\prime}$. The most important difference with respect to the definition of [46] is that we here express the parametric curve in terms of the general Legendre expansion (1.16), while in [46] authors assume the knowledge of $R_{\psi}$ at $\psi=\pi$.

Also in this case, expressions (1.48)-(1.50) are not easy to handle analytically. However, in the simplest case in which the expansion (1.16) has only two nonvanishing terms, such that $c_{0} \neq 0 \neq c_{1}$ and $c_{\ell}=0$, with $\ell \geq 2$, Eq. (1.43) takes compact form

$$
\begin{equation*}
2 c_{1} x^{2}+c_{0} x-c_{1}=0, \tag{1.53}
\end{equation*}
$$

with solution

$$
\begin{equation*}
x_{S}=-\frac{c_{0}}{4 c_{1}} \pm \sqrt{\frac{c_{0}^{2}}{16 c_{1}^{2}}+\frac{1}{2}}, \tag{1.54}
\end{equation*}
$$

and where the + or - signs refer to when $c_{1}>0$ and $c_{1}<0$, respectively. The corresponding quantities $R_{s, I I}, D_{s, I I I}$ and

$$
\begin{align*}
R_{s, \text { III }} & =\frac{2 c_{0}^{2}+c_{1}^{2}\left(1+x_{s}\right)+2 c_{0} c_{1}\left(1+x_{s}\right)}{2\left[c_{0}+c_{1}\left(1+x_{s}\right)\right]},  \tag{1.55}\\
D_{s, \text { III }} & =\frac{c_{1}^{2}\left(1+x_{s}\right)}{c_{0}+c_{1}\left(1+x_{S}\right)},  \tag{1.56}\\
\delta_{s, \text { III }} & =\frac{2 c_{1}^{2}\left(1+x_{s}\right)}{2 c_{0}^{2}+c_{1}^{2}\left(1+x_{s}\right)+2 c_{0} c_{1}\left(1+x_{s}\right)}=\delta_{1, \text { III }} . \tag{1.57}
\end{align*}
$$

If the shadow is perfectly circular with radius $c_{0}$, then $c_{l}=0$ and expressions (1.55) (1.57) show that $R_{s, I I I}=c_{0}, D_{s, I I I}=0=\delta_{s, I I I}$, as expected.

### 1.4. Application of the formalism to black-hole spacetimes

Having constructed a general formalism that allows us to describe in a coordinate independent manner the black-hole shadow and to measure its deformation, we are now ready to apply such a formalism to the specific case of some well-known spacetime metrics referring to axisymmetric black-holes. In particular, we will obviously start with the application of the formalism to a rotating (Kerr) black hole (in Sect. 1.4.1), to move over to a Bardeen black-hole and to a Kerr-Taub-NUT black hole in Sect. 1.4.2. We note that we do not consider these last two examples of black holes because they are particularly realistic, but simply because they offer analytic line elements on which our formalism can be applied.

### 1.4.1 Kerr black hole

We start with the Kerr spacetime, whose line element in Boyer-Lindquist coordinates reads

$$
\begin{align*}
d s^{2}= & -\left(1-\frac{2 M r}{\Sigma^{2}}\right) d t^{2}-\frac{4 a M r \sin ^{2} \theta}{\Sigma^{2}} d t d \varphi+\frac{\Sigma^{2}}{\Delta} d r^{2}+ \\
& +\Sigma^{2} d \theta^{2}+\left[\Sigma^{2}+\frac{a^{2}\left(\Sigma^{2}+2 M r\right) \sin ^{2} \theta}{\Sigma^{2}}\right] \sin ^{2} \theta d \varphi^{2}, \tag{1.58}
\end{align*}
$$

where

$$
\begin{equation*}
\Sigma:=r^{2}+a^{2} \cos ^{2} \theta, \quad \Delta:=r^{2}-2 M r+a^{2}, \tag{1.59}
\end{equation*}
$$

with $M$ being the mass of the black hole and $\mathrm{a}:=\mathrm{J} / \mathrm{M}$ its specific angular momentum.
Since the shape of the shadow is ultimately determined by the innermost unstable orbits of photons, hereafter we will concentrate on their equations for photons. In such a spacetime, the corresponding geodesic equations take the form

$$
\begin{align*}
\Sigma\left(\frac{q y}{q f}\right) & =\frac{\nabla}{\gamma E-J^{\top} \Gamma \Gamma^{\lrcorner} \cdot \Gamma^{2}}  \tag{1.60}\\
\Sigma^{2}\left(\frac{d r}{d \lambda}\right)^{2} & =\mathcal{R}  \tag{1.61}\\
\Sigma^{2}\left(\frac{d \theta}{d \lambda}\right)^{2} & =\Theta  \tag{1.62}\\
\Sigma\left(\frac{d \phi}{d \lambda}\right) & =\frac{2 a M r E}{\Delta}+\frac{(\Sigma-2 M r) L_{z}}{\Delta \sin ^{2} \theta}, \tag{1.63}
\end{align*}
$$

where $\lambda$ is an affine parameter,

$$
\begin{align*}
\mathcal{R} & :=E^{2} r^{4}+\left(a^{2} E^{2}-L_{z}^{2}-\mathcal{Q}\right) r^{2}+2 M\left[\left(a E-L_{z}\right)^{2}+\mathcal{Q}\right] r-a^{2} \mathcal{Q}  \tag{1.64}\\
\Theta & :=\mathcal{Q}\left(a^{2} E^{2}-L_{z}^{2} \csc ^{2} \theta\right) \cos ^{2} \theta  \tag{1.65}\\
A & :=\left(r^{2}+a^{2}\right)^{2}-a^{2} \Delta \sin ^{2} \theta \tag{1.66}
\end{align*}
$$

$E$ and $L_{z}$ are the photon's energy and angular momentum, respectively. The quantity Q

$$
\begin{equation*}
\mathcal{Q}=p_{\theta}^{2}+\cos ^{2} \theta\left(\frac{L_{z}^{2}}{\sin ^{2} \theta}-a^{2} E^{2}\right) \tag{1.67}
\end{equation*}
$$

is the so-called the Carter constant and $p_{\theta}:=\Sigma d \theta / d \lambda$ is the canonical momentum
conjugate to $\theta$. Using these definitions, it is possible to determine the unstable orbits as those satisfying the conditions

$$
\begin{equation*}
\mathcal{R}(\bar{r})=\frac{\partial \mathcal{R}(\bar{r})}{\partial r}=0, \quad \text { and } \quad \frac{\partial^{2} \mathcal{R}(\bar{r})}{\partial r^{2}}>0, \tag{1,68}
\end{equation*}
$$

where $\bar{r}$ is the radial coordinate of the unstable orbit. Introducing now the new parameters $\xi=L_{z} / E$ and $\eta=Q / E^{2}$, the celestial coordinates $\alpha$ and $\beta$ of the image plane of the distant observer are given by [88]

$$
\begin{equation*}
\alpha=\frac{\xi}{\sin i}, \quad \beta= \pm\left(\eta+a^{2} \cos ^{2} i-\xi^{2} \cot ^{2} i\right)^{1 / 2}, \tag{1.69}
\end{equation*}
$$

where $i$ is the inclination angle of the observer's plane, that is, the angle between the normal to the observer's plane and the black-hole's axis of rotation. In the case of the Kerr spacetime (1.58) and after using the conditions (1.68), one can easily find that the values of $\xi$ and $\eta$ relative to the circular orbit (c) are [34, 22]

$$
\begin{align*}
\xi_{c} & =\frac{M\left(\bar{r}^{2}-a^{2}\right)-\bar{r}\left(\bar{r}^{2}-2 M \bar{r}+a^{2}\right)}{a(\bar{r}-M)},  \tag{1.70}\\
\eta_{c} & =\frac{\bar{r}^{3}\left[4 a^{2} M-\bar{r}(\bar{r}-3 M)^{2}\right]}{a^{2}(\bar{r}-M)^{2}}, \tag{1.71}
\end{align*}
$$

Next, to investigate the shape of the black-hole shadow we introduce the generic celestial polar coordinates ( $R^{\prime}, \psi^{\prime}$ ) (cf. Sect. 1.2 and Fig. 1.1) defined as

$$
\begin{align*}
R^{\prime} & =\left(\alpha^{2}+\beta^{2}\right)^{1 / 2}  \tag{1.72}\\
\psi^{\prime} & =\tan ^{-1}\left(\frac{\beta}{\alpha}\right) \tag{1.73}
\end{align*}
$$

Assuming for simplicity that the observer is in the equatorial plane, i.e., that $i=\pi / 2$, then in terms of the $\left(R^{\prime}, \psi^{\prime}\right)$ coordinates the shadow of black hole can be described as

$$
\begin{align*}
& R^{\prime}=\frac{\left(2 r^{4}+2 a^{2} r-6 r^{2}+a^{2}+a^{2} r^{2}\right)^{1 / 2}}{r-1}  \tag{1.74}\\
& \psi^{\prime}=\tan ^{-1}\left(\frac{\left\{r^{3}\left[4 a^{2}-r(r-3)^{2}\right]\right\}^{1 / 2}}{r^{2}-a^{2}-r\left(r^{2}-2 r+a^{2}\right)}\right) \tag{1.75}
\end{align*}
$$

Making use of the procedure described in Sect. 1.2, it is straightforward to determine the coordinates (1.6)-(1.7) of the effective center of the shadow, and to express perform the Legendre expansion (1.16). To the best of our knowledge, no analytic expression exists to cast the coordinates (1.93)-(1.75) as a polar curve $R=R(\psi)$. However, such a curve can be easily constructed numerically and from it the Legendre expansion (1.16) can be computed.

Figure 1.4.
Left panel: Magnitude of the expansion coefficients $c_{\ell}$ of the polar curve and shown as a function of the expansion order. Note the very rapid (exponential) convergence of the expansion. The coefficients are computed for a Kerr black hole and different lines refer to different values of the spin parameter a. Right panel: Relative differences between the polar curve for the black-hole shadow as constructed from expressions (1.93)-(1.75) and the corresponding curve obtained from the expansion, i.e., $1-$ $\sum_{0}^{n} c_{\ell} P_{\ell} / R_{\psi}$. Different lines refer to different truncations of the expansion and show that three coefficients are sufficient to obtain deviations of a few percent.



Figure 1.5.
Magnitude as a function of the expansion order of the three different distortion parameters $\delta_{\mathrm{s}, \mathrm{I}}-\delta_{\mathrm{s}, \text { III }}$ defined as in Eqs. (1.32), (1.39), and (1.50), thus measuring the difference between the Legendre expanded polar curves $R_{\psi, I}-R_{\psi, I I}$ and the reference circles of radii $R_{\mathrm{s}, \mathrm{I}}-R_{\mathrm{s}, \text { III }}$. All curves refer to a Kerr black hole and different colours are used to represent different values of the spin parameter.


Figure 1.4 summarizes the results of our approach by reporting in the left panel and in a logarithmic scale the normalized values of the expansion coefficients $c_{\ell}$ as a function of the Legendre order $\ell$. Different curves refer to the different values considered for the spin parameter a, which ranges from $a=0.4$ (blue solid line) to $a=0.99$ (red solid line). Interestingly, the series converges very rapidly (essentially exponentially) and already with $\ell=4$, the contribution of higher-order terms is of the order of $10^{-2}$, decreasing further to $\sim 10^{-3}$ for $\ell=6$. Furthermore, even when considering the more severe test of $a=0.99$, the expansion coefficient with $\ell=6$ is only a factor twothree larger than the corresponding coefficient for a slowly rotating black hole. The right panel of Fig. 1.4 shows a direct measure of the relative differences between the polar curve for the black-hole shadow as constructed from expressions (1.93)-(1.75) and the corresponding curve obtained from the expansion, i.e., $1-\sum_{0}^{n} c_{\ell} P_{\ell} / R_{\psi}$. Remarkably, already when considering the first three terms in the expansion, i.e., $\mathrm{c}_{0}, \mathrm{c}_{1}$,
and $c_{2}$, the relative difference is of a few percent (blue line), and this further reduces to $10^{-3}$ when the expansion is truncated at $n=4$ (black line).

Figure 1.6.
Comparative view of the different distortion parameters $\delta_{\mathrm{s}, \mathrm{I}}$ (red line), $\delta_{\mathrm{s}, \mathrm{II}}$ (blue line), and $\delta_{s, \text { III }}$ (light-blue line). The left and right panels show the values of the distortion parameters as a function of the expansion order (cf. Fig. 1.5), and refer to a Kerr black hole with spin $a=0.40$ and $a=0.99$, respectively.


In summary, Fig. 1.4 demonstrates that when considering a Kerr black hole, the approach proposed here provides a coordinate independent and accurate representation of the black-hole shadow and that a handful of coefficients is sufficient for most practical purposes. In the following sections we will show that this is the case also for other axisymmetric black holes.

Before doing that, we show in Fig. 1.5 the values of the dimensionless distortion parameters as computed for the shadow of a Kerr black hole and for increasing values of the expansion index $\ell$. The three different panels are relative respectively to the parameters (1.32), (1.39), and (1.50), with the different curves referring to values of the
spin parameter a, ranging from $a=0.4$ (blue solid line) to $a=0.99$ (red solid line). As one would expect, for all values of $a$, each of the three distortion parameters decreases as the expansion includes higher-order terms. At the same time, because larger rotation rates introduce larger distortions in the shadow, they also lead to larger values of the distortion parameters for a fixed value of $\ell$.

Finally, Fig. 1.6 offers a comparative view of the different distortion parameters for specific values of the spin parameter, with the left and right panels referring to $a=0.4$ and $a=0.99$, respectively. This view is rather instructive as it shows that the different definitions lead to significantly different values of the distortion, despite they all refer to the same parametric polar curve. Furthermore, it helps appreciate that the distortion parameter $\delta_{s, I I}$ is systematically smaller than the other two and hence not the optimal one. This is because a larger value of the distortion parameter will increases the possibility of capturing the complex structure of the shadow. The fact that the curves for $\delta_{s, I}$ and $\delta_{s ; I I I}$ intersect for the Kerr black hole considered at $\ell=5$ implies that both distortion parameters (1.32) and (1.50) are useful indicators, with $\delta_{s, I I I}$ being the recommended choice for expansions having $\ell \geq 5$.

### 1.4.2 Bardeen and Kerr-Taub-NUT black holes

We continue our application of the formalism developed in Sects. 1.2 and 1.3 by considering the spacetime of a rotating Bardeen black hole [89]. We recall that in Boyer-Lindquist coordinates, the metric of a Kerr and of a Bardeen black hole differ uniquely in the mass, which needs to be modified as [37, 27]

$$
\begin{equation*}
M \rightarrow m=M\left(\frac{r^{2}}{r^{2}+g^{2}}\right)^{3 / 2} \tag{1.76}
\end{equation*}
$$

where the parameter $g$ is the magnetic charge of the nonlinear electrodynamic field responsible for the deviation away from the Kerr spacetime.

Figure 1.7.
Left panel: Magnitude of the expansion coefficients $c^{`}$ as a function of the expansion order $\ell$ for the different values of the magnetic charge of a Bardeen black hole: $g=0.00$ (red line), $g=0.30$ (blue line), and $g=0.50$ (light-blue line). All lines refer to a fixed value of the rotation parameter $a=0.60$ (cf. left panel of Fig. 1.4). Right panel: The same as in the left panel but for a Kerr-Taub-NUT black hole. Different curves refer to the different values of the NUT parameter: $n=0.00$ (red line), $n=0.30$ (blue line), and $n=0.60$ (light-blue line). All lines refer to a fixed value of the rotation parameter $\mathrm{a}=0.60$.



The impact parameters $\xi$ and $\eta$ relative to the circular orbit are in this case [27]

$$
\begin{align*}
& \xi_{c}=\frac{m\left[(2-f) \bar{r}^{2}-f a^{2}\right]-\bar{r}\left(\bar{r}^{2}-2 m \bar{r}+a^{2}\right)}{a(\bar{r}-f m)},  \tag{1.77}\\
& \eta_{c}=\frac{\bar{r}^{3}\left\{4(2-f) a^{2} m-\bar{r}[\bar{r}-(4-f) m]^{2}\right\}}{a^{2}(\bar{r}-f m)^{2}}, \tag{1.78}
\end{align*}
$$

and can be taken to define the shadow of black hole. Note that $m$ and $f$ are functions of the unstable circular radius $\bar{r}$ and are given by

$$
\begin{align*}
& m=m(\bar{r})=M\left(\frac{\bar{r}^{2}}{\bar{r}^{2}+g^{2}}\right)^{3 / 2},  \tag{1.79}\\
& f=f(\bar{r})=\frac{\bar{r}^{2}+4 g^{2}}{\bar{r}^{2}+g^{2}} . \tag{1.80}
\end{align*}
$$

In complete analogy, we can consider a Kerr-Taub-NUT black hole with nonvanishing gravitomagnetic charge $n$ and specific angular momentum $a:=J / M$. The corresponding metric is given by [69]

$$
\begin{align*}
d s^{2}= & -\frac{1}{\Sigma}\left(\Delta-a^{2} \sin ^{2} \theta\right) d t^{2}+\Sigma\left(\frac{d r^{2}}{\Delta}+d \theta^{2}\right)+\frac{1}{\Sigma}\left[(\Sigma+a \chi)^{2} \sin ^{2} \theta-\chi^{2} \Delta\right] d \phi^{2} \\
& +\frac{2}{\Sigma}\left(\Delta \chi-a(\Sigma+a \chi) \sin ^{2} \theta\right) d t d \phi, \tag{1.81}
\end{align*}
$$

where the functions $\Delta, \Sigma$ and $\chi$ are now defined as

$$
\begin{equation*}
\Delta:=r^{2}+a^{2}-n^{2}-2 M r, \Sigma:=r^{2}+(n+a \cos \theta)^{2}, \chi:=a \sin ^{2} \theta-2 n \cos \theta . \tag{1.82}
\end{equation*}
$$

In this case, the impact parameters $\xi$ and $\eta$ for the circular orbits are given by [36]

$$
\begin{equation*}
\xi_{c}=\frac{a^{2}(1+\bar{r})+\bar{r}^{2}(\bar{r}-3)+n^{2}(1-3 \bar{r})}{a(1-\bar{r})}, \tag{1.83}
\end{equation*}
$$

$$
\begin{equation*}
\eta_{c}=\frac{\bar{r}^{3}\left[4 a^{2}-\bar{r}(\bar{r}-3)^{2}\right]-n^{2}\left[4 \bar{r}^{2} a^{2}+(1-3 \bar{r})\left(n^{2}(1-3 \bar{r})-6 \bar{r}^{2}+4 a^{2} \bar{r}+2 \bar{r}^{3}\right)\right]}{a^{2}(\bar{r}-1)}, \tag{1.84}
\end{equation*}
$$

and define the shadow of the Kerr-Taub-NUT black hole.
Applying the formalism described in Sect. 1.2, it is possible to compute the coefficients of the Legendre expansion (1.16) also for a Bardeen and for a Kerr-TaubNUT black hole. The numerical values of these coefficients are reported as a function of the expansion order $\ell$ in Fig. 1.7, where the left panel refers to a Bardeen black hole, while the right one to a Kerr-Taub-NUT black hole. More specifically, the different lines in the left panel refer to different values of the magnetic charge: $g=0.00$ (red line), $g=0.30$ (blue line), and $g=0.60$ (light-blue line); all lines refer to a fixed value of the rotation parameter $a=0.60$. Very similar is also the content of the right panel of Fig. 1.7, which is however relative to a Kerr-Taub-NUT black hole. The different curves in this case refer to the different values of the NUT parameter: $n=0.00$ (red line), $n=0.30$ (blue line), and $n=0.60$ (light-blue line); once again, all lines refer to a fixed value of the rotation parameter $a=0.60$.

In analogy with what shown in the left panel of Fig. 1.4 for a Kerr black hole, also for these black holes the series converges very rapidly and already with $\ell=4$, the contribution of higher-order terms is of the order of $10^{-3}$, decreasing further to $\sim 10^{-5}$ for $\ell=6$, even when considering higher larger magnetic charges or NUT parameters. Furthermore, in analogy with the right panel of Fig. 1.4, we have checked that the relative differences between the polar curves for the shadow constructed from expressions (1.77)-(1.78) and (1.83)-(1.84), and the corresponding curve obtained from the expansion, i.e., $1-\sum_{0}^{n} c_{\ell} P_{\ell} / R_{\psi}$, is below $10^{-2}$ when $n=2$ (not shown in Fig. 1.7); this difference further reduces to $10^{-4}$ when the expansion is truncated at $\mathrm{n}=4$.

In conclusion, also Fig. 1.7 demonstrates that the approach proposed here provides a coordinate independent and accurate representation of black-hole shadows in spacetimes other than the Kerr one.

### 1.5. Comparison with noisy observational data

All of the considerations made so far have relied on the assumption that the shadow is a well-defined one-dimensional curve (cf. discussion in Sect 1.2). In practice, however, this is not going to be the case as the astronomical observations will have intrinsic uncertainties that, at least initially, will be rather large. It is therefore natural to ask how the formalism presented here will cope with such uncertainties. More precisely, it is natural to ask whether it will still be possible to determine the effective center of a noisy polar curve and then determine from there its properties. Although a very obvious and realistic problem, this concern is systematically ignored in the literature, where the shadow is traditionally assumed to have no uncertainty due to the observational measurements.

While awaiting for actual observational data, we can straightforwardly address this issue in our formalism and mimic the noisiness in the observational data by considering the polar curve as given by the Legendre expansion (1.16), where however the different coefficients $c_{\ell}$ are artificially perturbed. More specifically, we express the shadow via the new expansion

$$
\begin{equation*}
R_{\psi}=\sum_{\ell=0}^{\infty} c_{\ell}\left(1+\Delta_{c}\right) P_{\ell}(\cos \psi), \tag{1.85}
\end{equation*}
$$

where $\Delta_{c}$ is a random real number chosen uniformly in the range $\left[-\Delta_{\max } ; \Delta_{\max }\right.$ ]. In this way, our putative polar curve representing the shadow will be distorted following
a random distribution and we have optimistically assumed a variance of only $5 \%$, i.e., $\Delta_{\max }=0.05$. Of course, there is no reason to expect that the error distribution in the actual observational data will be uniform, but assuming a white noise is for us the simplest and less arbitrary choice.

With the setup described above and the formalism discussed in the previous Sections, we have considered a reference shadow of a Kerr black hole with spin parameter $a / M=0.99$ and have reproduced it after truncating the expansion (1.85) at $\ell=9$, which is more than sufficient given the accuracy obtained at this order (cf. Fig. 1.4). We have therefore constructed a very large number of such realizations of the observational shadow after making use of $N_{\text {tot }}=10^{5}$ draws of the random deformation $\Delta_{c}$. For each putative observed reconstructed shadow we have computed the distortion parameters $\delta_{s, I}-\delta_{s, I I I}$ defined in Eqs. (1.32), (1.39), and (1.50), as well as the distortion definition of [46] and defined in Eq. (1.51).

For each of the shadow realizations we have therefore computed the measurement error as

$$
\begin{equation*}
\epsilon_{*}:=\delta_{s}-\delta_{s, *}, \tag{1.86}
\end{equation*}
$$

where $\delta_{s}$ is the exact distortion of the background Kerr solution and measuring the relative difference of the shadow at $\psi=0$ and $\psi=\pi$. On the other hand, $\delta_{s, *}$ is given by either $\delta_{\mathrm{s}, \mathrm{I}}-\delta_{s, I I I}$ or $\delta_{\mathrm{s}, \mathrm{HM}}$.

Figure 1.8: Comparison of probability density distributions of the errors computed in the measurement of the distortion parameters $\delta_{s, I}-\delta_{s, I I}$ for a Kerr black-hole shadow reconstructed using a perturbed expansion [cf. Eq. (1.85)]. Also shown is the distribution of the error measured when using the distortion parameter introduced in [46] and that has a larger variance.


Figure 1.8 shows the distributions of the errors computed in this way for the four different possible definitions of the distortion parameters, with the black line referring to the distortion parameter in Eq. (1.51), and the red, blue and lightblue lines referring to the definitions (1.32), (1.39), and (1.50), respectively. Note that the values of the probability densities distributions are reported in such a way that

$$
\begin{equation*}
\frac{1}{N_{\text {tot }}} \int_{-\infty}^{\infty} d n=\frac{1}{N_{\text {tot }}} \int_{-\infty}^{\infty}\left(\frac{d n}{d \epsilon_{*}}\right) d \epsilon_{*}=1 . \tag{1.87}
\end{equation*}
$$

The distributions reported in Fig. 1.8 are rather self-explanatory. Clearly, all the different definitions are centered on $\varepsilon *=0$, indicating that on average they provide a good measurement of the distortion. On the other hand, the variance of the different distribution is rather different. Overall, the distortion parameters $\delta_{\mathrm{s}, \mathrm{I}}-\delta_{\mathrm{s}, \text { III }}$ have comparable variances, with a slightly smaller variations for the definition $\delta_{\mathrm{s}, \mathrm{I}}$. However, the variance of the distortion parameter for $\delta_{\mathrm{s}, \mathrm{HM}}$ is almost twice as large as the others and it essentially spans the $5 \%$ variation that we introduce in the random distortion $\Delta_{c}$. These results are rather reassuring as they indicate that new definitions are not only accurate, but also robust with respect to random white noise. Furthermore, they appear to be superior to other distortion measurements suggested in the past.

As a final remark we note that the introduction of the perturbations in the expansion (1.85) also has the effect of changing the position of the effective center of the shadow and hence the values of $\sim \overrightarrow{\boldsymbol{R}}_{0} 0$ and $\psi_{0}$ [cf. Eqs. (1.4) and (1.7)]. Fortunately, such variations represent only a high-order error, which is much smaller than those measured by the distortion parameters, with maximum measured variance of the order of $10^{-4}$. As a result, the distortions reported in Fig. 1.8 are genuine measurements of the shadow and not artefacts introduced by the changes in the effective centers.

### 1.6. Motion of the charged particles around rotating non-Kerr-black hole

The deformed Kerr-like metric which describes a stationary axisymmetric, and asymptotically flat vacuum spacetime, in the standard Boyer-Lindquist coordinates, can be expressed as [55]

$$
\begin{equation*}
d s^{2}=g_{00} d t^{2}+g_{11} d r^{2}+g_{22} d \theta^{2}+g_{33} d \phi^{2}+2 g_{03} d t d \phi, \tag{1.88}
\end{equation*}
$$

with

$$
\begin{aligned}
g_{00} & =-\left(1-\frac{2 M r}{\Sigma^{2}}\right)(1+h), \\
g_{11} & =\frac{\Sigma^{2}(1+h)}{\Delta+a^{2} h \sin ^{2} \theta}, \quad g_{22}=\Sigma^{2}, \\
g_{33} & =\sin ^{2} \theta\left[\Sigma^{2}+\frac{a^{2}\left(\Sigma^{2}+2 M r\right) \sin ^{2} \theta}{\Sigma^{2}}(1+h)\right], \\
g_{03} & =-\frac{2 a M r \sin ^{2} \theta}{\Sigma^{2}}(1+h),
\end{aligned}
$$

where

$$
\Sigma^{2}=r^{2}+a^{2} \cos ^{2} \theta, \quad \Delta=r^{2}-2 M r+a^{2}, \quad h=\frac{\epsilon M^{3} r}{\Sigma^{4}}
$$

and the constant $\varepsilon$ is the deformation parameter. The quantity $\varepsilon>0$ or $\varepsilon<0$ corresponds to the cases in which the compact object is more prolate or oblate than the Kerr black hole, respectively. As $\varepsilon=0$, the black hole is reduced to the typical Kerr black hole known in general relativity. The exact solutions of vacuum Maxwell equations in spacetime of the non-rotating black hole immersed in uniform magnetic field are presented in Appendix 1.

It is very important to study in detail the motion of charged particles around a rotating non-Kerr black hole immersed in a uniform magnetic field given by 4 -vector potential (A.1.3) with the aim to test the modified gravity theories.

Hereafter we will take into account that induced electric charge of the compact object will be rapidly neutralized due to the process of selective accretion of charges from the surrounding plasma. We shall study the motion of the charged test particles around a rotating non-Kerr black hole using the Hamilton-Jacobi equation

$$
\begin{equation*}
g^{\mu \nu}\left(\frac{\partial S}{\partial x^{\mu}}+e A_{\mu}\right)\left(\frac{\partial S}{\partial x^{\nu}}+e A_{\nu}\right)=-m^{2} \tag{1.89}
\end{equation*}
$$

where $e$ and $m$ are the charge and the mass of a test particle, respectively. Since $t$ and $\varphi$ are the Killing variables one can write the action in the form

$$
\begin{equation*}
S=-\mathcal{E} t+\mathcal{L} \varphi+S_{\mathrm{r} \theta}(r, \theta), \tag{1.90}
\end{equation*}
$$

where the conserved quantities $E$ and $L$ are the energy and the angular momentum of a test particle at infinity. Substituting it into equation (1.89) one can get the equation for inseparable part of the action.

One can easily separate variables in this equation in the equatorial plane $\theta=\pi / 2$ and obtain the equation for radial motion

$$
\begin{equation*}
\left(\frac{d \rho}{d s}\right)^{2}=\mathcal{E}^{2}-1-2 V_{\mathrm{eff}}^{2} \tag{1.91}
\end{equation*}
$$

where $s$ is the proper time along the trajectory of a particle, $\rho=r / M$, and

$$
\begin{align*}
V_{\mathrm{eff}}^{2}= & \frac{1}{8 \rho^{6}\left(\rho^{3}+\epsilon\right)^{2}}\left[16 a \mathcal{E} \mathcal{L} \rho^{6}\left(\rho^{3}+\epsilon\right)+a^{4} b^{2}(2+\rho)\left(\rho^{3}+\epsilon\right)^{3}\right. \\
& +\rho^{6}\left\{\rho^{5}\left(b^{2}(\rho-2) \rho^{2}-8\right)+\rho^{2}\left\{\rho\left(8 \mathcal{E}^{2}-4+b^{2}(\rho-2) \rho\right)-8\right\} \epsilon\right. \\
& \left.+4\left(\mathcal{E}^{2}-1\right) \epsilon^{2}+4 \mathcal{L}(\rho-2)\left(\rho^{3}+\epsilon\right)(b \rho+\mathcal{L})\right\}+2 a^{2} \rho^{4}\left(\rho^{3}+\epsilon\right) \\
& \left.\times\left\{\left[2+b\left\{2 \mathcal{L}+b\left(\rho^{2}-2\right)\right\}\right]\left(\rho^{3}+\epsilon\right)-2 \mathcal{E}^{2} \rho^{2}(2+\rho)\right\}\right] \tag{1.92}
\end{align*}
$$

can be interpreted as an effective potential of the radial motion, where $b=e B M / m$ is the magnetic parameter.

In the Fig. 1.9 the radial dependence of effective potential for radial motion of charged particles in the equatorial plane of rotating non-Kerr black hole immersed in
magnetic field is shown for the different values of magnetic parameter, deformation parameter and angular momentum of black hole. From this dependence one can obtain modification of radial motion of charged particle in the equatorial plane in the presence of the deformation parameter. One can obtain now how magnetic, rotational, and deformation parameters may change the character of the motion of the charged particles. Magnetic parameter is responsible for shifting of the shape of the effective potential towards the central black hole, which means that the minimum distance of the charged particles to the central object may decrease. As it is seen from the figure the deformation parameter changes the shape of the effective potential in the vicinity of the central object. This is caused by the appearance of the new term being proportional to the deformation parameter as $1 / r^{3}$ in the spacetime metric tensor. When the deformation parameter is positive $\varepsilon>0$, the shape of the effective potential shifts towards the central object which corresponds to the decrease of the radius of the circular orbits of the test particles. The opposite effect can bee observed when the deformation parameter is negative $\varepsilon<0$ and the graph shifts to the observer at infinity and corresponds to the increase of the radius of stable orbits. With the increase of the magnetic parameter and angular momentum of the central object the minimum of graphs shifts towards the central object. With the increase of the magnetic parameter the influence of the deformation parameter becomes weaker. The strong magnetic field dominates on behavior of the motion of charged particles with compare to the deformation parameter.

Figure 1.9.
The radial dependence of the effective potential of radial motion of charged particle around rotating non-Kerr black hole's equatorial plane. The figures a,b,c correspond to the case of $a=0$. The figures $\mathrm{d}, \mathrm{e}, \mathrm{f}$ corresponds to the case of $a=0.5$. The figures $\mathrm{g}, \mathrm{h}, \mathrm{i}$ correspond to the case of $a=0.98$. The figures $\mathrm{a}, \mathrm{d}, \mathrm{g}$ correspond to the case of $b=0$. The figures $\mathrm{b}, \mathrm{e}, \mathrm{h}$ correspond to the case of $\mathrm{b}=0: 05$. The figures $\mathrm{c}, \mathrm{f}, \mathrm{i}$ correspond to the case of $b=0.2$.


Figure 1.10.
The radial dependence of the effective potential of radial motion of charged particle around rotating non-Kerr black hole in equatorial plane. The figures correspond to the case of almost extreme black hole when $\mathrm{a}=0: 99$. The figure a corresponds to the case when $b=10^{3}$ and the figure b corresponds to the case when $b=10^{4}$.


In the recent paper [90, 87] it was shown that for protons and electrons the value of the dimensionless magnetic parameter is not weak which indicates that the effect of the magnetic field on a charged particle motion is not negligible. In general such magnetic field can essentially modify the motion of charged particles (for more details on estimation of magnetic parameter see e.g. [87]). Due to this reason now we will study the case when the dimensionless magnetic parameter $b \gg 1$.

The radial dependence of effective potential presented in Fig. 1.10 indicates the modification of radial motion of charged particle in the equatorial plane in the presence of the strong magnetic field and the deformation parameter. As above mentioned the magnetic parameter is responsible for the shift of the minimum distance of the charged particles towards the central object. The influence of the deformation parameter is strong in the vicinity of the central object which is due to the strong decay of the spacetime metric tensor as $1 / r^{3}$ with the increase of the radial coordinate. With the increase of the deformation parameter towards the positive values side the shape of the effective potential shifts towards the central object which corresponds to the decrease of the radius of the circular orbits of the test particles. The increase of the magnetic parameter shifts the effective potential upwards. The total energy of test particle will increase due to the increase of potential energy of interaction between the magnetic field and test charge. The strong magnetic field dominates on behavior of the motion of charged particles with compare to that of the deformation parameter.

### 1.7. Circular orbits around rotating non-Kerr black hole

In order to find solution for the ISCO radius $\mathrm{r}_{\text {ISCo }}$ we assume that the external magnetic field is absent. The expression (1.91) can now be written as

$$
\begin{aligned}
\left(\frac{d \rho}{d s}\right)^{2}= & f(\rho)=\frac{1}{\rho^{2}\left(\rho^{3}+\epsilon\right)}\left[2 \rho^{2}(a \mathcal{E}-\mathcal{L})^{2}+\rho^{3}\left(a^{2} \mathcal{E}^{2}-\mathcal{L}^{2}\right)-a^{2}\left(\rho^{3}+\epsilon\right)\right] \\
& +\frac{\rho^{2}}{\left(\rho^{3}+\epsilon\right)^{2}}\left[2 \rho^{3}+\left(\mathcal{E}^{2}-1\right) \rho^{4}-(\rho-2) \epsilon\right]
\end{aligned}
$$

First we will consider the case when $\varepsilon \ll 1$ in order to get approximate analytical solution for $r_{\text {ISCo }}$. Using the equation (1.93) and the condition of occurrence of circular orbits: $f(r)=0 ; f^{\prime}(r)=0$, one can easily find expressions for energy $E$ and angular momentum $L$ of a particle at circular orbit of radius $r_{\mathrm{c}}$, which are given as

$$
\begin{align*}
\mathcal{E}^{2} & =\frac{(\rho-2)^{2}}{\rho(\rho-3)}\left(1-\frac{\epsilon}{2 \rho^{2}(\rho-3)}\right)+\mathcal{O}\left(\epsilon^{2}\right),  \tag{1.93}\\
\mathcal{L}^{2} & =\frac{\rho^{2}}{\rho-3}\left(1-\frac{3(\rho-2)^{2} \epsilon}{2 \rho^{3}(\rho-3)}\right)+\mathcal{O}\left(\epsilon^{2}\right) . \tag{1.94}
\end{align*}
$$

Figure 1.11.
Radial dependence of energy (left plot) and angular momentum (right plot) of particle moving around the rotating non-Kerr black hole on circular orbits for the different values of deformation parameter when dimensionless rotational parameter $a / M=0.5$.


Fig 1.11 shows the radial dependence of both the energy and the angular momentum of the test particle moving on circular orbits around non-Kerr black hole in the equatorial plane. One can easily see that the presence of the negative deformation parameter $\varepsilon<0$ forces a test particle to have bigger energy and angular momentum in order to be kept on its circular orbit. It is a consequence of the increase of the gravitational potential of the rotating non-Kerr black hole with the negative deformation parameter. In the case of positive deformation parameter $\varepsilon>0$ the shape of graphs shifts towards the origin and means that the stable orbits shifts towards the central object.

For the expressions (1.93)-(1.94) one can easily find minimum radius for the circular orbits $\rho_{m c}=r_{m d} M$ as

$$
\begin{equation*}
\rho_{\mathrm{mc}}=3+\frac{\epsilon}{18}+\mathcal{O}\left(\epsilon^{2}\right) \tag{1.95}
\end{equation*}
$$

In the limiting case when $\varepsilon=0$ tends to zero $r_{\mathrm{mc}}=3 M$ which exactly coincides with the Schwarzschild limit. The minimum radius for a stable circular orbit will occur at the point of inflexion of the function $f(\rho)$, or in other words, we must supplement conditions $f(\rho)=f^{\prime}(\rho)$ together with the inequality $f^{\prime \prime}(\rho)<0$. The solution in the limit of small $\varepsilon$ has the following form

$$
\begin{equation*}
\rho_{\mathrm{ISCO}}=6-\frac{2 \epsilon}{9}+\mathcal{O}\left(\epsilon^{2}\right) . \tag{1.96}
\end{equation*}
$$

In the tables 1.1 and 1.2 we provide the numerical results on ISCO radius of charged particle around rotating non-Kerr black hole immersed in external magnetic field for the different values of deformation, rotation, and magnetic parameters. From the results, one can easily get in the case of the Schwarzschild spacetime $a=\varepsilon=b=0$ the
standard value for ISCO radius as $r_{\text {ISCO }}=6 \mathrm{M}$. With the increase of the deformation parameter $\varepsilon$ from -2 to 2 the radius of ISCO as well as the relative distance from the event horizon $\left(r_{\text {ISCo }}-r_{\mathrm{h}}\right) / r_{\mathrm{h}}$ monotonically decrease, where the radius of event horizon for the equatorial plane defined from $\Delta+a^{2} h=0$ (Table 1.1.). The presence of the magnetic field also decreases the radius of ISCO (Table 1.2.).

Table 1.1.
The innermost stable circular orbits and the value of the expression $\left(r_{\text {ISCo }}-r_{\mathrm{h}}\right) / r_{\mathrm{h}}$ of the particles moving around the rotating non-Kerr black hole (case of $b=0$ ).

| $\varepsilon$ | -2 | -1 | -0.5 | 0 | 0.5 | 1 | 2 |
| :--- | :--- | :--- | :--- | :--- | ---: | :--- | :--- |
| $a=0.0$ | 6.4345 | 6.22 | 6.1106 | 6.0 | 5.8884 | 5.776 | 5.5503 |
|  | 2.22 | 2.11 | 2.06 | 2.0 | 1.94 | 1.89 | 1.78 |
| $a=0.5$ | 4.9286 | 4.5947 | 4.4172 | 4.233 | 4.0426 | 3.848 | 3.460 |
|  | 1.58 | 1.43 | 1.35 | 1.27 | 1.18 | 1.09 | 0.90 |
| $a=0.7$ | 4.3383 | 3.9069 | 3.6625 | 3.3931 | 3.0933 | 2.7567 |  |
|  | 1.38 | 1.20 | 1.10 | 0.98 | 0.84 | 0.69 |  |
| $a=0.8$ | 4.0604 | 3.5629 | 3.2633 | 2.9066 | 2.4431 |  |  |
|  | 1.30 | 1.10 | 0.98 | 0.82 | 0.61 |  |  |
| $a=0.98$ | 3.6209 | 2.9849 | 2.5295 | 1.614 |  |  |  |
|  | 1.17 | 0.93 | 0.75 | 0.35 |  |  |  |

Now we will analyze ISCO in the astrophysical situation when $b \gg 1$. Using the expression for the effective potential (1.92) and the conditions $\frac{d \rho}{d s}=V^{\prime}{ }_{e f f}=V^{\prime \prime}{ }_{e f f}=$ 0 one can easily find the analytic expression for ISCO in the form

$$
\begin{equation*}
r_{\mathrm{ISCO}}=1+\frac{1-2 a^{2} / M^{2}-\epsilon}{\sqrt{6} b}+\mathcal{O}\left(b^{-2}, \epsilon^{2}\right) . \tag{1.97}
\end{equation*}
$$

The relation (1.97) shows the qualitative dependence of ISCO radius on the magnetic and the deformation parameters both. In the limit of strong magnetic interaction the magnetic field and the deformation parameter decrease the ISCO radius.

Table 1.2.
The innermost stable circular orbits and the value of the expression $\left(r_{\text {ISco }}-r_{h}\right) / r_{h}$ of the particles moving around the rotating non-Kerr black hole (case of $\mathrm{a}=0: 5$ ).

| $\varepsilon$ | -2 | -1 | -0.5 | 0 | 0.5 | 1 | 2 |
| :--- | :--- | :--- | :--- | :--- | ---: | :--- | :--- |
| $b=0.0$ | 4.9286 | 4.5947 | 4.4172 | 4.233 | 4.0426 | 3.848 | 3.460 |
|  | 1.58 | 1.43 | 1.35 | 1.27 | 1.18 | 1.09 | 0.90 |
| $b=0.05$ | 4.5857 | 4.3246 | 4.1828 | 4.0331 | 3.8758 | 3.7121 | 3.3941 |
|  | 1.41 | 1.29 | 1.23 | 1.16 | 1.09 | 1.01 | 0.87 |
| $b=0.2$ | 5.6081 | 5.5336 | 3.3261 | 3.2377 | 3.1527 | 3.0656 |  |
|  | 1.94 | 1.93 | 0.77 | 0.74 | 0.70 | 0.66 |  |

1.8. Charged particles motion around black hole with nonvanishing NUT charge

Here we will consider a charged particle motion in the vicinity of a black hole of mass $M$ with gravitomagnetic charge in the presence of an external axisymmetric and uniform at the spatial infinity magnetic field. The spacetime metric has form [91, 92]:

$$
\begin{align*}
d s^{2}= & -\frac{\Delta}{\Sigma} d t^{2}+\frac{4 \Delta}{\Sigma} l \cos \theta d t d \varphi+\frac{\Sigma^{2} \sin ^{2} \theta-4 \Delta l^{2} \cos ^{2} \theta}{\Sigma} d \varphi^{2} \\
& +\frac{\Sigma}{\Delta} d r^{2}+\Sigma d \theta^{2}, \tag{1.98}
\end{align*}
$$

where parameters $\Sigma$ and $\Delta$ are defined as

$$
\Sigma=r^{2}+l^{2}, \quad \Delta=r^{2}-2 M r-l^{2},
$$

where $l$ is the gravitomagnetic monopole momentum.
While considering the charged particle motion around black hole immersed in the magnetic field it is easy to use two conserved quantities associated with the Killing vectors: the energy $E>0$ and generalized azimuthal angular momentum $L \in(-\infty ; \infty)$ :

$$
\begin{align*}
\mathcal{E} & \equiv-\xi_{(t)}^{\mu} P_{\mu} \stackrel{m \Delta}{\Sigma}\left(\frac{d t}{d \tau}+4 l \cos \theta \frac{d \varphi}{d \tau}+\frac{q}{m} B l \cos \theta\right),  \tag{1.99}\\
\mathcal{L} & \equiv \xi_{(\phi)}^{\mu} P_{\mu} \\
& =-4 m l \frac{\Delta}{\Sigma} \cos \theta \frac{d t}{d \tau}+\left(\Sigma \sin ^{2} \theta-4 l^{2} \frac{\Delta}{\Sigma} \cos ^{2} \theta\right)\left(m \frac{d \varphi}{d \tau}+\frac{q B}{2}\right) . \tag{1.100}
\end{align*}
$$

Here $P_{\mu}=m u_{\mu}+q A_{\mu}$ is the generalized 4-momentum of a charged particle. It was first shown by [93] for spherical symmetric case (NUT spacetime) and later by [92] for axial symmetric case (Kerr-Taub-NUT spacetime) that the orbits of the test particles are confined to a cone with the opening angle $\theta$ given by $\cos \theta=2 E / / L$. It also follows that in this case the equations of motion on the cone depend on $l$ only via $l^{2}[92,77]$. The main point is that the small value for the upper limit for gravitomagnetic moment has been obtained by comparing theoretical results with experimental data as (i) $l \leq$ $10^{24}$ from the gravitational microlensing [94], (ii) $l \leq 1.5 \cdot 10^{18}$ from the interferometry experiments on ultra-cold atoms [74], (iii) and similar limit has been obtained from the experiments on Mach-Zehnder interferometer [65]. Due to the
smallness of the gravitomagnetic charge let us consider the motion in the quasiequatorial plane when the motion in $\theta$ direction changes as $\theta=\pi / 2+\delta \theta(t)$, where $\delta \theta(t)$ is the term of first order in $l$, then it is easy to expand the trigonometric functions as $\sin \theta=1-\delta \theta^{2}(t) / 2+O\left(\delta \theta^{4}(t)\right)$ and $\cos \theta=\delta \theta(t)-O\left(\delta \theta^{3}(t)\right)$. Neglecting the small terms $O\left(\delta \theta^{2}(t)\right)$, one can easily obtain the geodesic equation in the following form

$$
\begin{align*}
\frac{d t}{d \tau} & =\frac{\mathcal{E}}{m} \frac{\Sigma}{\Delta}  \tag{1.101}\\
\frac{d \varphi}{d \tau} & =\frac{\mathcal{L}}{m \Sigma}-\frac{q B}{2 m}  \tag{1.102}\\
\frac{d r}{d \tau} & =\left(\frac{\mathcal{E}}{m}-U\right), \tag{1.103}
\end{align*}
$$

where $U$ denotes the effective potential as

$$
\begin{equation*}
U=\frac{\Delta}{\Sigma}\left(1+\Sigma \chi^{2}\right), \tag{1.104}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi=\frac{\mathcal{L}}{m \Sigma}-\frac{q B}{2 m} . \tag{1.105}
\end{equation*}
$$

In the expressions (1.101)-(1.103) the terms being proportional to the second and higher orders of the small parameter $\delta \theta$ are neglected. In the Fig. 2.1 the radial dependence of the effective potential of the radial motion of the charged particle is presented for the different values of dimensionless gravitomagnetic parameter $\mathrm{l}=\mathrm{M}$ and magnetic parameter $\mathrm{b}=\mathrm{qBM}=\mathrm{m}$.

From the Fig. 1.12 one can obtain the behavior of the charged particle motion
in the presence of both the gravitomagnetic charge and magnetic field. In the presence of the gravitomagnetic charge the minimum of the effective potential shifts towards the observer at the infinity which means that the orbits of the charged particles may become unstable. The minimum value of the radius of the stable circular orbits increases. The influence of the magnetic field has the opposite effect: the presence of the magnetic field decreases the minimum value of the circular orbits radius and charged particles may come much closer to the central object.

Figure 1.12.
The radial dependence of the effective potential of radial motion of the charged particle for the different values of the gravitomagnetic charge (a): $l / M=0$ (solid line), $l / M=$ 0.4 (dot-dashed line), $l / M=0.8$ (dashed line) and for the different values of the dimensionless magnetic parameter $b=q B M / m(\mathrm{~b}): b=0.2$ (solid line), $b=0.4$ (dotdashed line), $b=0.8$ (dashed line).


In order to study innermost stable circular orbits (ISCO) we use its first and second derivatives of the $U$ and equalize them to zero:

$$
\begin{equation*}
U^{\prime}=-\frac{2 \Delta r(1+2 \mathcal{L} \chi)}{\Sigma^{2}}+\frac{1}{r}\left(1+\frac{\Delta}{\Sigma}\right)\left(1+\Sigma \chi^{2}\right)=0 \tag{1.106}
\end{equation*}
$$

$$
\begin{equation*}
U^{\prime \prime}=\frac{8 \Delta L^{2} r^{2}}{\Sigma^{4}}+\frac{2}{\Sigma}\left(\frac{4 \Delta r^{2}}{\Sigma^{2}}-\frac{3 \Delta}{\Sigma}-2\right)(1+2 \mathcal{L} \chi)++\frac{2}{\Sigma}\left(1+\Sigma \chi^{2}\right) \tag{1.107}
\end{equation*}
$$

Now we have two equations with three unknown quantities $L, r$ and $\chi$. Solving the equation (1.106) we derive $L$ in terms $r$ and $\chi$

$$
\begin{equation*}
\mathcal{L}=\frac{-2 \Delta r^{2}+\Delta \Sigma+\Sigma^{2}+\Delta \Sigma^{2} \chi^{2}+\Sigma^{3} \chi^{2}}{4 \Delta r^{2} \chi} . \tag{1.108}
\end{equation*}
$$

Substituting this equation into (1.107) one can easily obtain the equation expressing $\chi$ in terms of $r$.

$$
\begin{equation*}
\chi_{ \pm}= \pm\left[\frac{K}{\Sigma A}\left(1 \pm \sqrt{1-\frac{A \Sigma}{K^{2}}\left(\Delta+\Sigma-\frac{2 r^{2}}{\Sigma} \Delta\right)^{2}}\right)\right]^{\frac{1}{2}} \tag{1.109}
\end{equation*}
$$

where

$$
\begin{aligned}
& A=8 \Delta^{2} r^{2}-5 \Delta^{2} \Sigma+12 \Delta r^{2} \Sigma-8 \Delta \Sigma^{2}-3 \Sigma^{3} \\
& K=-2 \Delta^{2} r^{2}+2 \Delta^{2} \Sigma-4 \Delta r^{2} \Sigma+3 \Delta \Sigma^{2}+\Sigma^{3}
\end{aligned}
$$

The signs $\pm$ correspond to co-rotating and counter-rotating particle orbits, respectively. The dependence of the $\chi_{ \pm}$from the ISCO radius are shown in Fig. 1.13. The shift of the shape of + to the right direction with increasing the gravitomagnetic charge corresponds to increasing of ISCO radius in the presence of NUT charge.

From the Fig. 1.13 one can see that $\chi_{-}$function steadily increase when particle approach to the black hole horizon and go to infinity in the case when gravitomagnetic charge vanishes. In the case of nonvanishing NUT-parameter, the function $\chi_{-}$falls in
the point of singularity. However, by reason of that jumps are located inside the horizon, the effects that take place there are not observable relatively to detached observer and cannot be interpreted as some full physical theory. In other hand the spacetime of nonrotating black hole allows us to find analytic continuations of our theories inside the black hole horizon up to a point of singularity by excepting it.

Figure 1.13.
$\chi_{+}$(a) and $\chi_{-}$(b) as a function of the ISCO radius for different values of the gravitomagnetic charge: $l / M=0$ (solid line), $l / M=0.4$ (dot-dashed line), $l / M=0.8$ (dashed line).


We concentrate on a circular motion of the charged particle in the presence of NUT charge. At ISCO radius the effective potential has minimum. The 4-momentum of a test charged particle at the circular orbit of radius $r$ is ([87])

$$
\begin{align*}
p^{\mu} & =m \gamma\left(e_{(t)}^{\mu}+v e_{(\phi)}^{\mu}\right)  \tag{1.110}\\
e_{(t)}^{\mu} & =(\Sigma / \Delta)^{1 / 2} \xi_{(t)}^{\mu}=(\Sigma / \Delta)^{1 / 2} \delta_{t}^{\mu},  \tag{1.111}\\
e_{(\phi)}^{\mu} & =\Sigma^{-1 / 2} \xi_{(\phi)}^{\mu}=\Sigma^{-1 / 2} \delta_{\phi}^{\mu} . \tag{1.112}
\end{align*}
$$

Here $v$ is a velocity of a charged particle with respect to a rest frame. $v$ can be both positive and negative, and $\gamma$ is the Lorentz gamma factor, which is always positive. Using the condition of the normalization for the momentum $\boldsymbol{p}^{2}=-m^{2}$ one has $\gamma=$ $\left(1-v^{2}\right)^{-1 / 2}$. For the positive charge $q>0$ the Lorentz force acting on a charged particle with $v>0$ is repulsive, i.e. the force is directed outwards the black hole, while for $v<0$ the Lorenz force is attractive.

Following to [87], one can use an expression $\frac{d \phi}{d \tau}=v \gamma / r$ and (1.102) with $\theta=\pi / 2$. This implies

$$
\begin{equation*}
\frac{v \gamma}{r}=\chi . \tag{1.113}
\end{equation*}
$$

From this expression one can easily find

$$
\begin{equation*}
\gamma=\sqrt{1+r^{2} \chi^{2}} \text { and } v=\frac{r \chi}{\sqrt{1+r^{2} \chi^{2}}} . \tag{1.114}
\end{equation*}
$$

Using (1.114) one can find the values of the velocity $v_{ \pm}$and $\gamma$-factor. Fig. 1.14 shows the velocity of a particle at the ISCO as a function of its radius, while Fig. 1.15 shows the dependence of $\gamma_{ \pm}$from $r_{I S C O}$.

The Fig. 1.14 shows the dependence of the velocity of the particle at ISCO. Since there are two values of velocity for the same radius one can interpret them as two values of velocity which correspond for two opposite directions of motion of the particles. Since charged particle and magnetic field interaction depends on the velocity direction there should be two values $v_{ \pm}$for each $r_{I S C O}$. Furthermore, the absence of external magnetic field (right border of the plots) one can obtain only one solution for the
velocity at ISCO. One should mention that in the nonrelativistic case one can get the Keplerian velocity profile.

Figure 1.14.
Velocity of the particles at $r_{I S C O}$ as a function of its radius for different values of the gravitomagnetic charge: $l / M=0$ (solid line), $l / M=0.4$ (dot-dashed line), $l / M=$ 0.8 (dashed line).


As the next step, following to [87] we will study the center-of-mass collision of two particles in the vicinity of a black hole with nonvanishing gravitomagneticcharge, when one of these particles has the mass $m$ and charge $q$ and rotates along the circular orbit. Another particle, which is neutral has a mass $\mu$ and 4-momentum $k$ and freely falls from the rest at spatial infinity. From the conservation of momenta one can write the momentum of the system at the moment of collision as

$$
\begin{equation*}
\mathrm{P}=\mathrm{p}+\mathrm{k} \tag{1.115}
\end{equation*}
$$

Figure 1.15.
$\gamma_{+}$(a) and $\gamma_{-}$(b) as a function of the ISCO radius for different values of the gravitomagnetic charge: $l / M=0$ (solid line), $l / M=0.4$ (dot-dashed line), $l / M=0.8$ (dashed line).


This implyies that the center-of-mass energy $E_{c . m}$ : of two colliding particles is

$$
\begin{equation*}
E_{\text {c.m. }}=m^{2}+\mu^{2}-2(\mathbf{p}, \mathbf{k}) . \tag{1.116}
\end{equation*}
$$

Using the equation of particle motion around black hole with nonvanishing gravitomagnetic charge (1.101)-(1.103) one can obtain the following relation for the center-of-mass energy of two colliding particles for the weak magnetic field approximation:

$$
\begin{equation*}
\frac{E_{\text {c.m. }}}{m} \simeq 0.3 \sqrt{\frac{96-l^{2}}{\sqrt{8+l^{2}}}} b^{1 / 4} . \tag{1.117}
\end{equation*}
$$

In the Table 1.3 the dependence of the ISCO radius and the center of mass energy of colliding charged particles have peen shown. From the results on can conclude that gravitomagnetic charge correction prevents the particle from the infinite acceleration.

Table 1.3.
The dependence of the center of mass energy and ISCO radii from the magnetic parameter b for the different values of the specific gravitomagnetic charge $l / M$ :

| $l / M$ | 0 | 0.4 | 0.8 | 1.0 | 4.52 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $r_{I S C O}$ | $2+0.58 b^{-1}$ | $2.08+0.59 b^{-1}$ | $2.28+0.61 b^{-1}$ | $2.41+0.62 b^{-1}$ | $5.63+0.74 b^{-1}$ |
| $E_{c . m} / m$ | $1.747 b^{1 / 4}$ | $1.738 b^{1 / 4}$ | $1.708 b^{1 / 4}$ | $1.688 b^{1 / 4}$ | $1.129 b^{1 / 4}$ |

From the obtained result one can observe that the presence of the gravitomagnetic charge will decrease the value of the center of mass energy. The role of the magnetic field in particle accelerating process is to decrease the innermost stable circular orbits radii. As particles come closer to the black hole horizon their energy at infinity is going to increase. The role of the gravitomagnetic charge in this process is opposite: the presence of the gravitomagnetic charge increase the radii of ISCO.

### 1.9. Conclusion

The radio-astronomical observations of the shadow of a black hole would provide convincing evidence about the existence of black holes. Further, the study of the shadow could be used to learn about extreme gravity near the event horizon and determine the correct theory of gravity in a regime that has not been explored directly so far. A number of different mathematical descriptions of the shadow have already been proposed, but all make use of a number of simplifying assumptions that are unlikely to be offered by the real observational data, e.g., the ability of knowing with precision the location of the center of the shadow.

To remove these assumptions we have developed a new general and coordinateindependent formalism in which the shadow is described as an arbitrary polar curve expressed in terms of a Legendre expansion. Our formalism does not presume any
knowledge of the properties of the shadow and offers a number of routes to characterize the properties of the curve. Because our definition of the shadow is straightforward and unambiguous, it can be implemented by different teams analyzing the same noisy data.

The Legendre expansion used in our approach converges exponentially rapidly and we have verified that a number of coefficients less than ten is sufficient to reproduce the shadow with a precision of one part in $10^{5}$, both in the case of a Kerr black hole with spin parameter of $\mathrm{a} / \mathrm{M}=0.99$, and in the case of Bardeen and Kerr-Taub-NUT black holes with large magnetic charges and NUT parameters. Furthermore, the use of a simple Legendre expansion has allowed us to introduce three different definitions of the distortion of the shadow relative to some reference circles. The comparison of the different definitions has allowed us to determine which of them is best suited to capture the complex structure of the shadow and its distortions.

Finally, again exploiting the advantages of the Legendre expansion, we were able to simulate rather simply the presence of observational random errors in the measurements of the shadow. Constructing a large number of synthetically perturbed shadows, we have compared the abilities of the different parameters to measure the distortion in the more realistic case of a noisy shadow. Overall, we have found that our new definitions have error distributions with comparable variances, but also that these are about a factor of two smaller than the corresponding variance measured when using more traditional definitions of the distortion. Given these results, the approach proposed here could be used in all those studies characterizing the shadows of black holes as well as in the analysis of the data from experimental efforts such as EHT and BHC.

Analytic expressions for the vacuum electromagnetic fields external to rotating non-Kerr black hole embedded in asymptotically uniform magnetic field are presented. Derived exact expressions (A.1.6)-(A.1.9) for the vacuum electromagnetic field in the vicinity of the spacetime of rotating non-Kerr black hole indicate that electromagnetic field will be affected by the deformation parameter. However, the induced electric field
(A.1.6)-(A.1.7) depends on the deformation parameter $\varepsilon$ linearly while the magnetic field (A.1.8)-(A.1.9) depends on $\varepsilon$ quadratically.

Further motion of charged particles around rotating non-Kerr black hole immersed in external uniform magnetic field have been investigated using the Hamilton- Jacobi equations of motion. We have shown that the magnetic parameter $b$ being responsible for the interaction between magnetic field and charged particles shifts the minimum of the effective potential towards the central object, which means that the minimum distance of the charged particles to the central object decreases. The deformation parameter changes the shape of the effective potentials near to the central object. This is caused by the appearance of the new term being proportional to the deformation parameter as $1 / r^{3}$ in the spacetime metric. When the deformation parameter is positive the decrease of the radius of the circular orbits can be observed. The opposite effect can be observed when the deformation parameter is negative, i.e. when $\varepsilon<0$ the increase of the radius of stable orbits takes place.

We have studied in detail the influence of magnetic, rotational and deformation parameters on ISCO radius of charged particles around rotating non-Kerr black hole. The increase of the magnetic field and angular momentum of black hole decreases the radius of the stable circular orbits. While the deformation parameter is negative the ISCO radius of test particles becomes bigger than one for the undeformed case. For the positive values of the deformation parameter the ISCO radius decreases.

The recent measurements of the ISCO radius in accretion disks around black holes may also give alternate constraints on the numerical values of the deformation parameter. All astrophysical quantities related to the observable properties of the accretion disk can be obtained from the black hole spacetime metric and observations in the near infrared or X-ray bands have provided important information about the spin of the black holes [95, 96, 97]. It was stated that rotating black holes have spins in the
range $0.5 \leq a \leq 1$ that is according to the observations ISCO radii are essentially shifted towards the central objects.

Because of the spacetime structure the negative deformation parameter present some important differences with respect to their disc accretion properties as compared to the standard general relativistic Schwarzschild and Kerr ones. Therefore the study of the innermost stable orbits in the vicinity of compact objects is a powerful indicator of their physical nature. Since the ISCO radius decreases in the case of increase of the deformation parameter in the non-Kerr black holes one may compare these effects with the standard general relativistic ones. Finally since there is correlation between deformation parameter and stable orbits around black holes one may numerically calculate the upper limit of deformation parameter corresponding to the observable ISCO radius. One can put the observable values of ISCO radius into the inequality $V^{\prime \prime}{ }_{e f f}(r)<0$ and numerically solve it with respect to the deformation parameter $\varepsilon$ in order to get an upper limit for the deformation parameter as $\varepsilon \leq 22$.

In this chapter we have obtained the expressions for the energy and angular momentum as well as ISCO of the charged particle in the vicinity of the black hole in presence of gravitomagnetic charge and exterior magnetic field.

Recently, [84] underlined that a rotating black hole can, in principle, accelerate the particles falling to the central black hole to arbitrary high energies. [87] has shown that the magnetic field could play a role of charged particle accelerator near the nonrotating black hole. Because of some mechanisms such as astrophysical limitations on the maximum spin, backreaction effects, upper limit for magnetic field, and sensitivity to the initial conditions, there appears to be some upper limit for the center of mass energy of the infalling particles. One of the mechanisms offered in this work is appearing due to the gravitomagnetic charge correction which prevents the particle from the infinite acceleration.

# CHAPTER II. OPTICAL PROPERTIES OF BLACK HOLE IN THE PRESENCE OF PLASMA 

### 2.1. Introduction

The study of astrophysical processes in plasma medium surrounding black hole becomes very interesting and important due to the evidence for the presence of black holes at the centres of the galaxies [98, 28, 99]. For example, the gravitational lensing in inhomogeneous and homogeneous plasma around black holes has been recently studied in [100, 101, 102, 103, 104] as extension of vacuum studies (see, e.g. [43, 105]).

From the literature it is known that the black hole shadow is appeared by the gravitational lensing effect, see, e.g. [39, 27, 42, 28]. If black hole is placed between a bright source and far observer, dark zone is created in the source image by photon fall inside black hole which is commonly called shadow of black hole. Recently, this effect is investigated by many authors for the different black holes (see, e.g. [46, 44, 38, 106]). The silhouette shape of an extremely rotating black hole has been investigated by Bardeen [18]. Our previous studies on the shadow of black hole are related to the nonKerr [42], Ho rava-Lifshitz [106], Kerr-Taub-NUT [36] and Myers-Perry [107] black holes. A new coordinate-independent formalism for characterization of a black-hole shadow has been recently developed in [14].

Shape of black hole is determined through boundary of the shadow which can be studied by application of the null geodesic equations. The presence of plasma in the vicinity of black holes changes the equations of motion of photons which may lead to the modification of black hole shadow by the influence of plasma. In this chapter our main goal is to consider silhouette of shadow of axially symmetric black hole using the equations of motion for photons in plasma with radial power-law density. We would like to underline that very recently, influence of a non-magnetized cold plasma with the
radially dependent density to black hole shadow has been studied in [103] using the different alternate approach. In addition Author of the Ref. [108] has studied the photon motion around black hole surrounded by plasma.

The chapter is based on the paper [15] of the author and arranged as following. In Sect. 2.2, we consider the equations of motion of photons around axially symmetric black hole in the presence of plasma. In Sect. 2.3 we study the shadow of the axialsymmetric black hole in the presence of plasma. As particular case in subsections 2.3.1 and 2.3.2 we study the shadow and the energy emission from the spherically symmetric black hole. Finally, in Sect. 3.9 we briefly summarize the results found.
2.2. Photon motion around the black hole in the presence of plasma

The rotating black hole is described by the space-time metric, which in the standard Boyer-Lindquist coordinates, can be written in the form

$$
\begin{equation*}
d s^{2}=g_{\alpha \beta} d x^{\alpha} d x^{\beta} \tag{2.1}
\end{equation*}
$$

with [109]

$$
\begin{align*}
& g_{00}=-\left(1-\frac{2 M r}{\Sigma}\right), \\
& g_{11}=\frac{\Sigma}{\Delta}, \\
& g_{22}=\Sigma, \\
& g_{33}=\left[\left(r^{2}+a^{2}\right)+\frac{2 a^{2} M r \sin ^{2} \theta}{\Sigma}\right] \sin ^{2} \theta, \\
& g_{03}=-\frac{2 M a r \sin ^{2} \theta}{\Sigma}, \\
& \Delta=r^{2}+a^{2}-2 M r, \quad \Sigma=r^{2}+a^{2} \cos ^{2} \theta, \tag{2.2}
\end{align*}
$$

where as usual $M$ and $a$ are the total mass and the spin parameter of the black hole.
Here we will consider the plasma surrounding the central axially symmetric black hole. The refraction index of the plasma will be $n=n\left(x^{i}, \omega\right)$ where the photon frequency measured by observer with velocity $u^{\alpha}$ is $\omega$. In this case the effective energy of photon has the form $\hbar \omega=-p_{\alpha} u^{\alpha}$. The refraction index of the plasma as a function of the photon four-momentum has been obtained in [110] and has the following form:

$$
\begin{equation*}
n^{2}=1+\frac{p_{\alpha} p^{\alpha}}{\left(p_{\beta} u^{\beta}\right)^{2}}, \tag{2.3}
\end{equation*}
$$

and for the vacuum case one has the relation $n=1$. The Hamiltonian for the photon around an arbitrary black hole surrounded by plasma has the following form

$$
\begin{equation*}
H\left(x^{\alpha}, p_{\alpha}\right)=\frac{1}{2}\left[g^{\alpha \beta} p_{\alpha} p_{\beta}+\left(n^{2}-1\right)\left(p_{\beta} u^{\beta}\right)^{2}\right]=0 . \tag{2.4}
\end{equation*}
$$

Following to the derivation of a gravitational redshift discussed in [111] we will assume that the spacetime stationarity allows existence of a timelike Killing vector $\xi^{\alpha}$ obeying to the Killing equations

$$
\begin{equation*}
\xi_{\alpha ; \beta}+\xi_{\beta ; \alpha}=0 . \tag{2.5}
\end{equation*}
$$

Then one can introduce two frequencies of electromagnetic waves using null wavevector $k^{\alpha}$ the first one is the frequency measured by an observer with four velocity $u^{\alpha}$ and defined as

$$
\begin{equation*}
\omega \equiv-k^{\alpha} u_{\alpha} \tag{2.6}
\end{equation*}
$$

while the second one is the frequency associated with the timelike Killing vector $\xi^{\alpha}$ and defined as

$$
\begin{equation*}
\omega_{\xi} \equiv-k^{\alpha} \xi_{\alpha} . \tag{2.7}
\end{equation*}
$$

The frequency (2.6) depends on the observer chosen and is therefore a function of position, while the frequency (2.7) is a conserved quantity that remains unchanged along the trajectory followed by the electromagnetic wave. One can apply this property to measure how the frequency changes with the radial position and is redshifted in the spacetime. Assume the Killing vector to have components

$$
\begin{equation*}
\xi^{\alpha} \equiv(1,0,0,0) ; \quad \xi_{\alpha} \equiv g_{00}(-1,0,0,0) \tag{2.8}
\end{equation*}
$$

so that $\omega_{\xi}=k_{0}=$ const. The frequency of an electromagnetic wave emitted at radial position $r$ and measured by an observer with four-velocity $u_{\alpha}\left\{\frac{1}{\sqrt{-g_{00}}}, 0,0,0\right\}$ parallel to $\xi^{\alpha}$ (i.e. a static observer) will be governed by the following equation

$$
\begin{equation*}
\sqrt{-g_{00}} \omega(r)=\omega_{\xi}=\text { const } . \tag{2.9}
\end{equation*}
$$

One may introduce a specific form for the plasma frequency for analytic processing, assuming that the refractive index has the general form

$$
\begin{equation*}
n^{2}=1-\frac{\omega_{e}^{2}}{\omega^{2}} \tag{2.10}
\end{equation*}
$$

where $\omega_{e}$ is usually called plasma frequency. Now using the Hamilton-Jacobi equation which defines the equation of motion of the photons for a given spacetime geometry [100,108,110]:

$$
\begin{equation*}
\frac{\partial S}{\partial \sigma}=-\frac{1}{2}\left[g^{\alpha \beta} p_{\alpha} p_{\beta}-\left(n^{2}-1\right)\left(p_{0} \sqrt{-g^{00}}\right)^{2}\right], \tag{2.11}
\end{equation*}
$$

where $p_{\alpha}=\partial S / \partial x^{\alpha}$. Using method of separation of variables the Jacobi action $S$ can be written as [109, 42]:

$$
\begin{equation*}
S=\frac{1}{2} m^{2} \sigma-\mathcal{E} t+\mathcal{L} \phi+S_{r}(r)+S_{\theta}(\theta) \tag{2.12}
\end{equation*}
$$

where $L, E$ are conservative quantities as angular momentum and energy of particles.
For trajectories of the photons we have the following set of the equations:

$$
\begin{align*}
\Sigma \frac{d t}{d \sigma}= & a\left(\mathcal{L}-n^{2} \mathcal{E} a \sin ^{2} \theta\right) \\
& +\frac{r^{2}+a^{2}}{\Lambda}\left[\left(r^{2}+a^{2}\right) n^{2} \mathcal{E}-a \mathcal{L}\right],  \tag{2.13}\\
\Sigma \frac{d \phi}{d \sigma}= & \left(\frac{\mathcal{L}}{\sin ^{2} \theta}-a \mathcal{E}\right)+\frac{a}{\Delta}\left[\left(r^{2}+a^{2}\right) \mathcal{E}-a \mathcal{L}\right],  \tag{2.14}\\
\Sigma \frac{d r}{d \sigma}= & \sqrt{\mathcal{R}},  \tag{2.15}\\
\Sigma \frac{d \theta}{d \sigma}= & \sqrt{\Theta}, \tag{2.16}
\end{align*}
$$

can be derived from the Hamilton-Jacobi equation, where the functions $R(r)$ and $\Theta(\theta)$ are introduced as

$$
\begin{align*}
\mathcal{R}= & {\left[\left(r^{2}+a^{2}\right) \mathcal{E}-a \mathcal{L}\right]^{2}+\left(r^{2}+a^{2}\right)^{2}\left(n^{2}-1\right) \mathcal{E}^{2} } \\
& -\Delta\left[\mathcal{K}+(\mathcal{L}-a \mathcal{E})^{2}\right]  \tag{2.17}\\
\Theta= & \mathcal{K}+\cos ^{2} \theta\left(a^{2} \mathcal{E}^{2}-\frac{\mathcal{L}^{2}}{\sin ^{2} \theta}\right) \\
& -\left(n^{2}-1\right) a^{2} \mathcal{E}^{2} \sin ^{2} \theta \tag{2.18}
\end{align*}
$$

and the Carter constant as $K$.
For calculation examples one needs the analytical expression of the plasma frequency $\omega_{e}$ which for the electron plasma has the following form

$$
\begin{equation*}
\omega_{e}^{2}=\frac{4 \pi e^{2} N(r)}{m_{e}} \tag{2.19}
\end{equation*}
$$

where $e$ and $m_{e}$ are the electron charge and mass respectively, and $N(r)$ is the plasma number density. Following the Ref. [108] here we consider a radial power-law density

$$
\begin{equation*}
N(r)=\frac{N_{0}}{r^{h}}, \tag{2.20}
\end{equation*}
$$

where $h \geq 0$, such that

$$
\begin{equation*}
\omega_{e}^{2}=\frac{k}{r^{h}} . \tag{2.21}
\end{equation*}
$$

As an example here we get the value for power $h$ as 1 [108]. For this value we plot the radial dependence of effective potential $V_{\text {eff }}$ of radial motion of the photons defined as

$$
\begin{equation*}
\left(\frac{d r}{d \sigma}\right)^{2}+V_{\mathrm{eff}}=1 \tag{2.22}
\end{equation*}
$$

The radial dependence of the effective potential for different values of plasma refraction $n$ and black hole spin a has been presented in Fig. 2.1. In the Fig.2.1 the left plot corresponds to the case when refraction parameter of the plasma is $n^{2}=$ 0.2; 0.44; 0.89 (dotted, dashed and solid lines, respectively) at the position $r=3 M$; middle plot corresponds to the case when the refraction parameter is $n^{2}=$ 0.19; $0.42 ; 0.88$ corresponding to dotted, dashed and solid lines, respectively, at the position $r=3 M$; right plot represents the radial dependence of the effective potential when refraction parameter is $n^{2}=0.14 ; 0.39 ; 0.88$ corresponding to dotted, dashed and solid lines, respectively, at the position $r=3 M$.

Figure 2.1.
The radial dependence of the effective potential of radial motion of photons for the different values of rotation parameter and refraction index of the plasma. Here the quantity $V_{\text {eff }}$ is normalised by the energy of the photon E.



2.3. Shadow of black hole in the presence of the plasma

In this section we consider the shadow cast by black hole surrounded by plasma. If black hole surrounded by plasma originated between the light source and the observer, then the latter can observe the black spot on the bright background. The observer at the infinity can only observe the light beam scattered away and due to capturing of the photons by the black hole the shaded area on the sky would be appeared. This spot corresponds to the shadow of the black hole and its boundary can be defined using the equation of motion of photons given by expressions (2.13)-(2.16) around black hole surrounded by plasma.

In order to describe the apparent shape of the the black hole surrounded by plasma we need to consider the closed orbits around it. Since the equations of motion depend on conserved quantities $E, L$ and the Carter constant $K$, it is convenient to parametrize them using the normalised parameters $\xi=L / E$ and $\eta=K / E^{2}$. The silhouette of the black hole shadow in the presence of the plasma can be found using the conditions

$$
\mathcal{R}(r)=0=\partial \mathcal{R}(r) / \partial r
$$

Using these equations one can easily find the expressions for the parameters $\xi$ and $\eta$ in the form

$$
\begin{align*}
\xi= & \frac{\mathcal{B}}{\mathcal{A}}+\sqrt{\frac{\mathcal{B}^{2}}{\mathcal{A}^{2}}-\frac{\mathcal{C}}{\mathcal{A}}}  \tag{2.23}\\
\eta= & \frac{\left(r^{2}+a^{2}-a \xi\right)^{2}+\left(r^{2}+a^{2}\right)^{2}\left(n^{2}-1\right)}{\Delta} \\
& -(\xi-a)^{2} \tag{2.24}
\end{align*}
$$

where we have used the following notations

$$
\begin{align*}
\mathcal{A}= & \frac{a^{2}}{\Delta}  \tag{2.25}\\
\mathcal{B}= & \frac{a^{2}-r^{2}}{M-r} \frac{M a}{\Delta}  \tag{2.26}\\
\mathcal{C}= & n^{2} \frac{\left(r^{2}+a^{2}\right)^{2}}{\Delta} \\
& +\frac{2 r\left(r^{2}+a^{2}\right) n^{2}+\left(r^{2}+a^{2}\right)^{2} n n^{\prime}}{M-r} \tag{2.27}
\end{align*}
$$

and prime denotes the differentiation with respect to radial coordinate $r$.
The boundary of the black hole's shadow can be fully determined through the expessions (2.23)-(2.24). However, the shadow will be observed at 'observer's sky', which can be referenced by the celestial coordinates related to the real astronomical measurements. The celestial coordinates are defined as

$$
\begin{align*}
\alpha & =\lim _{r_{0} \rightarrow \infty}\left(-r_{0}^{2} \sin \theta_{0} \frac{d \phi}{d r}\right),  \tag{2.28}\\
\beta & =\lim _{r_{0} \rightarrow \infty} r_{0}^{2} \frac{d \theta}{d r} . \tag{2.29}
\end{align*}
$$

Using the equations of motion (2.13)-(2.16) one can easily find the relations for the celestial coordinates in the form

$$
\begin{equation*}
\alpha=-\frac{\xi}{n \sin \theta}, \tag{2.30}
\end{equation*}
$$

$$
\begin{equation*}
\beta=\frac{\sqrt{\eta+a^{2}-n^{2} a^{2} \sin ^{2} \theta-\xi^{2} \cot ^{2} \theta}}{n}, \tag{2.31}
\end{equation*}
$$

for the case when black hole is surrounded by plasma.
In Fig 2.2 the shadow of the rotating black hole for the different values of black hole rotation parameter a, inclination angle 0 between the observer and the axis of the rotation is represented. In this figures we choose the plasma frequency in the form $\omega_{e} / \omega_{\xi}=k / r$. From the Fig. 2.2 one can observe the change of the size and shape of the rotating black hole surrounded by plasma. Physical reason for this is due to gravitational redshift of photons in the gravitational field of the black hole. The frequency change due to gravitational redshift affects on the plasma refraction index.

### 2.3.1 Shadow of non-rotating black hole

Now in order to extract pure plasma effects we will concentrate at the special case when the black hole is non-rotating and the size of the black hole shadow can be observed (see, e.g. [103]). In the case of the static black hole the shape of the black hole is circle and the radius of the shadow will be changed by plasma effects. Using the expressions (2.30) and (2.31) one can easily find the radius of the shadow of static black hole surrounded by plasma in the form:

$$
\begin{align*}
R_{s h}= & \frac{1}{n(r-M)}\left[2 r^{3}(r-M) n^{2}+r^{4} n n^{\prime}(r-M)\right. \\
& -2 r^{2} M^{2}+2 M r^{2}\left\{n r^{2}\left(n+r n^{\prime}\right)-\left(4 n+3 r n^{\prime}\right)\right. \\
& \left.\left.\times n M r+M^{2}\left(1+3 n^{2}+2 r n n^{\prime}\right)\right\}^{1 / 2}\right]^{1 / 2}, \tag{2.32}
\end{align*}
$$

Figure 2.2.
The shadow of the black hole surrounded by plasma for the different values of the rotation parameter a, inclination ante between observer and the axis of the rotation $\theta_{0}$, and the refraction index. The solid lines in the plots correspond to the vacuum case, while for dashed lines we choose the plasma frequency $\omega_{e} / \omega_{\xi}=k / r$ and $(k / M)^{2}=0.5$.

here $r$ is the unstable circular orbits of photons defined by $d r / d \sigma=0$ and $\partial V_{e f f} / \partial r=$ 0 . In the absence of the plasma one has the standard value of the photon sphere radius as $r=3 M$ and for shadow radius $R_{s h}=3 \sqrt{3}[112,113]$. In the presence of the plasma we will have different value for the photon sphere radius and consequently different shadow radius of the boundary of black hole shadow. In the Fig. 2.3 the dependence of the radius of shadow of the static black hole from the plasma parameters has been presented which shows that the radius of the shadow of black hole surrounded by inhomogeneous plasma decreases. It is similar to the results of the paper [103].

### 2.3.2 Emission energy of black holes in plasma

For the completeness of our study here we evaluate rate of the energy emission from the black hole in plasma using the expression for the Hawking radiation at the frequency $\Omega$ as [114, 115]

$$
\begin{equation*}
\frac{d^{2} E(\Omega)}{d \Omega d t}=\frac{2 \pi^{2} \sigma_{l i m}}{\exp \Omega / T-1} \Omega^{3}, \tag{2.33}
\end{equation*}
$$

where $T=\kappa / 2 \pi$ is the Hawking temperature, and $\kappa$ is the surface gravity. Here, for simplicity we consider the special case when the black hole is non rotating, and the background spacetime is spherically-symmetric.

At the horizon the temperature $T$ of the black hole is

$$
\begin{equation*}
T=\frac{1}{4 \pi r_{+}} . \tag{2.34}
\end{equation*}
$$

The limiting constant $\sigma_{\text {lim }}$

$$
\sigma_{l i m} \approx \pi R_{s h}^{2}
$$

Figure 2.3.
The dependence of the Schwarzschild black hole shadow on the plasma frequentcey parameter. Here we take the plasma frequency as $\omega_{e} / \omega_{\xi}=k / r$.

defines the value of the absorption cross section vibration for spherically symmetric black hole and $R_{s h}$ is given by expression (2.32). Consequently, one can get

$$
\frac{d^{2} E(\Omega)}{d \Omega d t}=\frac{2 \pi^{3} R_{s h}^{2}}{e^{\Omega / T}-1} \Omega^{3}
$$

that the energy of radiation of black hole in plasma depends on the size of its shadow.

The dependence of energy emission rate from frequency for the different values of plasma parameters $\omega_{e}$ is shown in Fig. 2.4. One can see that with the increasing plasma parameter $\omega_{e}$ the maximum value of energy emission rate decreases, caused by radius of shadow decrease.

### 2.4 Conclusions

Here we have studied shadow and emission rate of axial symmetric black hole in presence of plasma with radial power-law density. The obtained results can be summarized as follows.

- In the presence of plasma the observed shape and size of shadow changes depending on i) plasma parameters, ii) black hole spin and iii) inclination angle between observer plane and axis of rotation of black hole.
- In order to extract pure effect of plasma influence on black hole image the particular case of the Schwarzschild black hole has also been investigated. It is shown that under influence of plasma the observed size of shadow of the spherical symmetric black hole becomes smaller than that in the vacuum case. So it has been shown that i) the photon sphere around the spherical symmetric black hole is left unchanged under the plasma influence, ii) however the Schwarzschild black hole shadow size in plasma is reduced due to the refraction of the electromagnetic radiation in plasma environment of black hole.
- The study of the energy emission from the black hole in plasma has shown that with the increase of the dimensionless plasma parameter the maximum value of energy emission rate from the black hole decreases due to the decrease of the size of black hole shadow.

Figure 2.4.
Energy emission from black hole for the different values of $k / M$ : Solid line corresponds to the vacuum case $(k / M=0)$, Dashed line corresponds to the case when $k / M=0.4$, dot-dashed line corresponds to the case when $k / M=0.6$, dotted line corresponds to the case when $k / M=0.8$. Here we take the plasma frequency as $\omega_{e} / \omega_{\xi}=k / r$ and $\tilde{\Omega}$ is normalized to the $T$.


# CHAPTER III. ENERGETIC PROCESSES AROUND BLACK HOLE IN HO`RAVA-LIFSHITZ GRAVITY 

### 3.1. Introduction

Collisions of particles freely falling from rest at infinity can give extremely high Centre-of-Mass (CM) energy, if they occur in a very close vicinity of the black hole horizon of extreme Kerr black holes with dimensionless spin $a=M$, as shown in [84]. Nevertheless, BSW process needs a very fine-tuning of the motion constants of the colliding particles and imply some doubts on reality of such processes [116, 117, 118, 119]. Moreover, it has been shown that due to the gravitational redshift effect, the energy of the ultra-relativistic particles created in the BSW processes has to be comparable to the rest energy of the colliding particles, if observed by distant observers. The energy efficiency of the BSW process is thus substantially reduced by the gravitational redshift and is close to the rest energy of the colliding particles [116, 120]. An exceptional situation is possible if the electromagnetic interaction can be relevant in the collisional process [121, 6, 9].

In the field of near-extreme superspinning Kerr geometries, i.e., Kerr naked singularities or primordial Kerr superspinars [67, 122, 123], in the final stages of their conversion to near-extreme black holes due to accretion processes [124], the extremely high CM energy can be obtained with no fine tuning of the motion constants. It has been demonstrated for particles freely falling from infinity in the equatorial plane [125, 126] or along "radial" trajectories with arbitrary latitude [124, 127, 128, 129], if they collide at (or near) $r=M$, and for collisions of particles moving in the equatorial plane along the stable circular orbits located at $r=M$ with any particle freely falling from infinity [130]. It can also be obtained in generic collisions of particles freely falling
from infinity with covariant energy $E=m$ assuming that the collisions occur just at the radius $r=M$ [131].

The efficiency of escaping of the created highly-energetic particles and the energyconversion efficiency relative to distant observers due to the frequency shift of the highenergy photons (ultra-relativistic particles) produced in the collisions are the crucial phenomena related to the observational relevance of the ultra-high energy collisions. In the field of black holes, both these efficiencies are restricted by the gravitational redshift effect as demonstrated in [116]. On the other hand, both these efficiencies can remain large in the field of near-extreme superspinning Kerr geometry, if the collisions occur at $r=M$ and close enough to the equatorial plane of the geometry [131]. For such an effect, both the non-existence of the black hole horizon and the strong rotation of the superspinning near-extreme Kerr geometry are probably relevant, as the efficiencies decrease substantially, if the particles collide near the symmetry axis of the Kerr geometry [131]. To clear up the situation, a study of the collisions of the same kind occurring in a spherically symmetric naked singularity spacetime has to be realized. Here we demonstrate the relevance of this statement by considering the phenomenon of the ultra-high-energy collisions in the field of near-extreme Kehagias-Sfetsos naked singularities that represent an interesting spherically symmetric solution of the modified Horava gravity [132].

The present study of the acceleration process in the KS naked singularity spacetime is complementary to the recent works related to the particle acceleration mechanism in the field of 5-dimensional Kerr black holes [8] and black strings [9]. Moreover, the combined influence of the brane tension and the cosmological constant on the acceleration process has been considered in [121], while the acceleration of charged particle near the black holes with non vanishing gravitomagnetic charge has been studied in [6].

The Horava (or Horava-Lifshitz) gravity [133, 134, 135, 136, 137, 138, 139, 140] is recently considered as one of the promising approaches to the quantum gravity, being inspired by solid-body physics, namely the Lifshitz theory. The Horava gravity breaks the Lorentz invariance at the high-energy limit, while at the low-energy limit it reduces to the General Relativity and satisfies the Lorentz invariance. The solutions of the Horava effective gravitational equations have been found in [141, 142]. The spherically symmetric solution having asymptotically the Schwarzschild character has been found in the framework of the modified Horava model - the solution is described by the so called Kehagias-Sfetsos (KS) metric [132, 143, 144], which allows for existence of both black hole and naked singularity spacetimes. Slowly rotating black hole solutions of the modified Horava gravity has been found in [145, 146].

In connection to the accretion phenomena, the KS metric describing black holes has been extensively studied in a series of works related both to the particle motion [147, 146, 148, 149, 150, 151, 152] and optical phenomena [39, 153, 106] that can be relevant for tests of validity of the Horava gravity. The modified Ho rava model has been also tested for the properties of the magnetic field near spherical stars [152].

The properties of the circular geodesics of the KS naked singularity spacetimes have been shown to be similar to those related to the well known spherically symmetric naked singularity spacetimes, namely the Reissner-Nordstrom [154] and braneworld naked singularity spacetimes [155, 156, 157, 43, 158, 159], allowing for occurence of principally new astrophysical phenomena.

Recently Horava proposed a UV (Ultra-Violet) complete, non-relativistic gravity theory which is power-counting renormalizable one giving up the Lorentz invariance [160, 133]. Since then, many authors paid attention to this scenario to apply it to the black hole $(\mathrm{BH})$ physics $[161,162,144,163,164]$, cosmology [165, 166, 167, 168, 169, 170, 171] and observational tests [169]. Here we investigate the Penrose process around rotating BHs in the Horava-Lifshitz gravity theory. The quantum interference
effects [147] and the motion of the test particle around BH [148] in Horava-Lifshitz gravity have been also recently studied.

Recently authors of the paper [150] have studied the particle motion in the spacetime of a KS black hole. Potentially observable properties of black holes in the deformed Horava-Lifshitz gravity with Minkowski vacuum: the gravitational lensing and quasinormal modes have been studied in [144]. The authors of the paper [172] derived the full set of equations of motion, and then obtained spherically symmetric solutions for UV completed theory of Einstein proposed by Horava.

Black hole solutions and the full spectrum of spherically symmetric solutions in the five-dimensional nonprojectable Horava-Lifshitz type gravity theories have been recently studied in [173]. Geodesic stability and the spectrum of entropy/area for black hole in Horava-Lifshitz gravity via quasi-normal modes approach are analyzed in [174]. Particle geodesics around Kehagias-Sfetsos black hole in Horava-Lifshitz gravity are also investigated by authors of the paper [175]. Recently observational constraints on Horava-Lifshitz gravity have been found from the cosmological data [176]. Authors of the paper [161] have found all spherical black hole solutions for two, four and six derivative terms in the presence of Cotton tensor.

Recently the rotating black hole solution in the context of the Horava-Lifshitz gravity has been obtained in [161]. Here we study the energy extraction mechanism through the Penrose process and particle acceleration mechanisms near the rotating black hole in the Horava-Lifshitz gravity. We concentrate on the particles freely falling from rest at infinity and colliding in the deep gravitational field of the KS naked singularities, searching for conditions allowing for occurrence of the ultra-high CM energy collisions. In analogy with our studies of these phenomena in the superspinning Kerr geometries, we expect the ultra-high-energy collisions to occur near the surface $r=M$, in the field of near-extreme KS spacetimes. Then we test the efficiency of the acceleration process for charged particles following circular orbits, assuming the KS
naked singularities immersed in an asymptotically uniform magnetic field. We would like to stress that we consider high-energy collisions that still allow us to apply the Horava gravity in its General Relativistic limit, i.e., we consider motion of the particles and photons along geodesics of the spacetime.

The results and methodology used in this chapter is based on the following papers $[3,8,9,10,11,12,13]$ of the author and organized as follows: the geometry of the Kehagias-Sfetsos spacetime is analyzed in Sect. 3.2. Ultra-high energy collision of the particles are studied in Sect. 3.3. Motion of charged test particles in the field of KS naked singularities immersed into asymptotically uniform magnetic field are studied in Sect. 3.4. Collisions of charged particles moving along circular orbits in the presence of external magnetic field are considered in Sect. 3.5. The description of the rotating black hole solution and ergosphere around it considered in the Sect. 3.6. Penrose process in the ergosphere of the rotating black hole in Horava-Lifshitz gravity has been studied in Sect. 3.7. Sect. 3.8 is devoted to study the particle acceleration mechanism near the black hole in Horava-Lifshitz gravity. We conclude our results in Sect. 3.9.

### 3.2. Kehagias-Sfetsos spacetime

### 3.2.1. Geometry

The spherically symmetric solution of the so called modified Horava gravity, allowing for the Schwarzschild spacetime as an appropriate limit, is the KehagiasSfetsos (KS) spacetime [132], described in the standard Schwarzschild coordinates and the geometric units by the line element

$$
\begin{equation*}
d s^{2}=-f(r) d t^{2}+f^{-1}(r) d r^{2}+r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \varphi^{2}\right) \tag{3.1}
\end{equation*}
$$

where

$$
\begin{equation*}
f(r)=1+r^{2} \omega\left[1-\left(1+\frac{4 M}{\omega r^{3}}\right)^{1 / 2}\right] . \tag{3.2}
\end{equation*}
$$

In terms of $x=r / M$,

$$
\begin{equation*}
f(x)=1+x^{2}\left(\omega M^{2}\right)\left[1-\left(1+\frac{4}{\omega M^{2} x^{3}}\right)^{1 / 2}\right] . \tag{3.3}
\end{equation*}
$$

The pseudosingularities of the geometry, the horizons, are located at

$$
\begin{equation*}
r_{ \pm}=M \pm \sqrt{M^{2}-\frac{1}{2 \omega}} . \tag{3.4}
\end{equation*}
$$

Two horizons of the KS black hole spacetimes exist, if

$$
\begin{equation*}
\omega \geq \omega_{h}=\frac{1}{2 M^{2}} . \tag{3.5}
\end{equation*}
$$

The limit of the Schwarzschild black hole is obtained when $\omega \rightarrow \infty$. The horizon coincide when $\omega=\omega_{h}$, giving an extreme KS black hole spacetime. The KS naked singularity spacetimes occur, if

$$
\begin{equation*}
\omega<\omega_{h} . \tag{3.6}
\end{equation*}
$$

Recall that KS spacetimes are not Ricci flat - in fact, in the Schwarzschild limit, as the parameter $\omega \rightarrow \infty$, the Ricci scalar

$$
\begin{equation*}
R \sim \frac{1}{\omega} . \tag{3.7}
\end{equation*}
$$

For simplicity we can put $M=1$, i.e., we express all the quantities in the dimensionless form, as the dimensionality is governed by the mass parameter $M$ of the KS solution of the modified Horava gravity. The parameter $\omega$ ("Horava parameter") then governs modification of the gravitational law in the modified Horava gravitational theory.

### 3.2.2 Geodesic equations

The motion of test particles and photons is assumed to be governed by geodesics of the spacetime. The geodesic equation for the 4 -momentum $p^{\mu}$ of test particles (photons)

$$
\begin{equation*}
\frac{D p^{\mu}}{d \lambda}=0 \tag{3.8}
\end{equation*}
$$

can be in the spherically symmetric KS spacetimes separated and integrated easily. Due to the axial symmetry and stationarity of the KS spacetimes two constants of motion arise:

$$
\begin{equation*}
p_{\phi}=L, p_{t}=-E \tag{3.9}
\end{equation*}
$$

that are identified with the axial angular momentum $L$ and covariant energy $E$ related to the distant static observers. In spherically symmetric spacetimes, the motion occurs in the central planes; for a single particle motion, the plane can be chosen to be the equatorial plane. Considering motion in a general central plane, an additional motion constant, L, corresponding to the total angular momentum of the test particle (photon),
can be introduced. The geodesic equations in the integrated and separated form take then the form (see, e.g., [177])

$$
\begin{align*}
p^{t} & =g^{t t} p_{t}=\frac{E}{f(r ; \omega)},  \tag{3.10}\\
p^{\phi} & =g^{\phi \phi} p_{\phi}=\frac{L}{r^{2}},  \tag{3.11}\\
{\left[p^{\theta}\right]^{2} } & =\frac{1}{r^{4}}\left[\mathcal{L}^{2}-\frac{L^{2}}{\sin ^{2} \theta}\right],  \tag{3.12}\\
{\left[p^{r}\right]^{2} } & =E^{2}-f(r ; \omega)\left(m^{2}+\frac{\mathcal{L}^{2}}{r^{2}}\right), \tag{3.13}
\end{align*}
$$

where $m$ is the rest energy (mass) of the particle that is also a motion constant; for photons there is $m=0$. In the analysis of massive-particle or massless photon motion, it is usefull to use the notion of the effective potential that can be expressed in the form

$$
\begin{equation*}
V_{e f f}=f(r ; \omega)\left(m^{2}+\frac{\mathcal{L}^{2}}{r^{2}}\right) . \tag{3.14}
\end{equation*}
$$

For the motion in the equatorial plane $(\theta=\pi / 2)$, there is $p^{\theta}=0$.
The photon motion, the Keplerian orbits in KS spacetime, escaping cones and frequency shift of photons generated at the static reference systems are presented in [11].
3.3. Ultra-high CM energy of colliding particles around black hole in HoravaLifshitz gravity

We calculate the CM energy of colliding particles in three characteristic cases. Two of them correspond to the situations when the ultra-high-energy collisions were treated in the field of near-extreme superspinning Kerr geometry [124, 67, 131] namely the collisions of particles in radial free fall from infinity ( $E=m$ and $L=0$ ) with particles following geodesic circular orbits, or with radially escaping particles having $E=m$ and $L=0$ that inverted their motion at $r=0$. As the third case we consider collisions of particles moving in the equatorial plane with opposite angular momentum at the turning point of their radial motion. For simplicity, we consider collisions of identical particles, having the same rest energy.

Let the 4 -momenta of the colliding particles are $p_{1}^{\mu}$ and $p_{2}^{\mu}$ with total 4momentum $p_{\text {tot }}^{\mu}=p_{1}^{\mu}+p_{1}^{\mu}$. The corresponding CM energy then reads

$$
\begin{equation*}
E_{C M}^{2}=-p_{\text {tot }} \cdot p_{\text {tot }}=2 m^{2}\left(1-u_{1} \cdot u_{2}\right)=2 m^{2}\left(1-g_{\alpha \beta} u_{1}^{\alpha} u_{2}^{\beta}\right) . \tag{3.15}
\end{equation*}
$$

3.3.1. Collisions of particles on circular geodesics with radially falling particles

The circular orbits are assumed to be located at the equatorial plane. We assume for simplicity the particle falling from infinity also in the equatorial plane. The 4velocity of the particle on the circular geodesic is then given by

$$
\begin{equation*}
u_{1}^{\alpha}=\left(\frac{E_{K} / m}{f}, 0,0, \frac{L_{K} / m}{r^{2}}\right), \tag{3.16}
\end{equation*}
$$

where the constants of the motion on the circular orbits are the specific energy $E_{K}(r ; \omega) / m$ and specific angular momentum $L_{K}(r ; \omega) / m$. For a particle radially falling from infinity with covariant energy $E / m=1$ and zero angular momentum, the 4 -velocity reads

$$
\begin{equation*}
u_{1}^{\alpha}=\left(\frac{1}{f(r, \omega)},-\sqrt{1-f(r, \omega)}, 0,0\right) . \tag{3.17}
\end{equation*}
$$

The CM energy of the colliding particles reads

$$
\begin{equation*}
\frac{E_{C M}^{2}}{4 m^{2}}=\frac{1}{2}\left[1+\frac{E_{K}}{f}\right] . \tag{3.18}
\end{equation*}
$$

The CM energy is normed to the rest energy of the colliding particles that equals $2 m$. The energy of the test particle moving along a circular orbit at a radius $r$ is given by the relation (A.2.14). The behaviour of the CM energy occurring at this kind of collisions is presented in Fig. 3.1. One clearly sees that the maximum of the collisional CM energy decreases with decreasing value of parameter $\omega$. The location of the maximum shifts to the larger values of the radius $r$ with decreasing value of the parameter $\omega$.

For the stable circular orbits corresponding to matter in accretion discs extended to large distance from the KS naked singularity, the center-of-mass energy remains finite for all values of the Horava parameter ! corresponding to KS naked singularities. However, in the field of KS naked singularities with $1 / 2>\omega>0.3849$, an additional, inner accretion disc can occur that can extend from the inner edge at the static radius $r_{\text {stat }}=1 /(2 \omega)^{1 / 3}$ where particles with $L=0$ are located, to the outer edge corresponding to the inner photon circular orbit at $r_{p h 2}$ that is stable relative to the radial
perturbations. As the circular orbit at the inner disc is approaching the outer edge, both the specific energy and the specific angular momentum of the particle diverge (Fig. 3.2). Then also the CM energy of the colliding particles diverges; moreover, it is also influenced by the metric factor $1 / f(r ; \omega)$ that is very large in the near-extreme KS naked singularity spacetimes in vicinity of the specific radius $r=M$. Unfortunately, it is unclear, how to obtain by natural physical processes the ultra-relativistic particles following the stable circular orbits. Some ultra-relativistic particles following the stable circular orbits could be created during preceding collisional processes, however, their covariant energy corresponds to the rest energy of the freely infalling particles.

Figure 3.1.
The plots of the CM energy of the collisions between radially falling particles and orbiting particles on Keplerian orbits are given for five representative values of the parameter $\omega=0.1,0.15,0.2,0.281$, and 0.3 . The dashed line represents $E_{C M}$ at the static radius $r_{\text {stat }}=1 /(2 \omega)^{1 / 3}$. The dotted line indicates the $E_{C M}$ for collisions that appear in the region of unstable Keplerian orbits.


Figure 3.2.
Plot of the radial profiles of the specific energy $E_{K}^{2}$ (specific angular momentum $L_{K}^{2}$ ) of the Keplerian orbits on the left (right) panel for two representative values of parameter $\omega=0.36$ and 0.4 when both the outer and inner discs are allowed. The inner disc is constituted only by stable Keplerian orbits. In the outer disc, the inernal part corresponds to the unstable Keplerian orbits that are represented by the dotted part of the curve.


3.3.2 Collisions of particles with opposite angular momenta

Let two particles fall from infinity with covariant energy $E_{1,2}=m$ with opposite angular momenta $L_{1}=-L_{2}=L$. Assuming an opposite angular momentum of colliding particles, both particles have to move in the same central plane, and it is enough to consider only the equatorial motion to describe the problem in full generality. The components of the 4 -velocities for both particles read

$$
\begin{align*}
& u_{1}^{\mu}=\left(\frac{1}{f}, \sqrt{1-f(r)\left(1+\frac{L^{2} / m^{2}}{r^{2}}\right)}, 0,-\frac{L / m}{r^{2}}\right)  \tag{3.19}\\
& u_{2}^{\mu}=\left(\frac{1}{f}, \sqrt{1-f(r)\left(1+\frac{L^{2} / m^{2}}{r^{2}}\right)}, 0,+\frac{L / m}{r^{2}}\right) . \tag{3.20}
\end{align*}
$$

The square of the CM energy then reads

$$
\begin{equation*}
\frac{E_{C M}^{2}}{4 m^{2}}=1+\frac{L^{2} / m^{2}}{r^{2}} . \tag{3.21}
\end{equation*}
$$

When we let such particles to collide at the turning point $r_{t}=r_{t}(\omega,|L|)$ of their radial motion, which is given by the equation $1-f(r)\left(1-L^{2} / r^{2}\right)=0$, the specific angular momentum of the colliding particles at a given $r$ is then given by

$$
\begin{equation*}
L_{t}^{2}(r ; \omega)=r^{2}\left(\frac{1}{f(r ; \omega)}-1\right) \tag{3.22}
\end{equation*}
$$

giving the square of the CM energy in the form

$$
\begin{equation*}
\frac{E_{C M}^{2}}{4 m^{2}}=\frac{1}{f\left(r_{t}, \omega\right)} \tag{3.23}
\end{equation*}
$$

The behaviour of the CM energy $E_{C M}^{2} / 4 m^{2}$ as the function of the radius $r$ and the parameter $\omega$ is illustrated in Figs 3.3.

The discontinuity in the curves plotted for parameter $\omega=0.35,0.37$, and 0.4 is present due to the existence of the circular Keplerian orbits with covariant energy $E=m$. This kind of circular orbits exists for $\omega \geq \omega_{c}=0.327764$. The maxima of $E_{C M}^{2}$ radial profiles for this kind of collisions increase with increasing value of parameter $\omega$. The corresponding angular momentum shifts to larger values with increasing value of $\omega$.

Now, we turn our attention to the case of the collisions at the radius $r=M$, and in the near-extreme KS naked singularity spacetimes with the Horava parameter

$$
\begin{equation*}
\omega=\frac{1}{2}-\delta, \delta \ll 1 \tag{3.24}
\end{equation*}
$$

Figure 3.3.
The plots of the radial profile of the CM energy of collisions of particles with angular momenta $L_{1}=-L_{2}=L$ and covariant energy $E_{1} / m=E_{2} / m=1$. The plots are constructed for eight representative values of the parameter $\omega=0.1,0.2,0.25,0.28$ (left hand side figure), $0.3,0.35,0.37$, and 0.4 (right-handside figure).



Then the metric coefficient $f(r=M, \omega=1 / 2-\delta) \sim 2 \delta / 3$, and the CM energy takes the value of

$$
\begin{equation*}
\frac{E_{C M}^{2}}{4 m^{2}} \sim \frac{3}{2 \delta} \tag{3.25}
\end{equation*}
$$

that diverges as $\omega \rightarrow 1 / 2$ - this a similar situation as those occuring in the Kerr naked singularity spacetimes [67, 131]. In this special case of the ultra-high-energy collisions, the angular momentum of the colliding particles has to take very large value.

$$
\begin{equation*}
L_{H E}^{2} \sim \frac{3}{2 \delta} \tag{3.26}
\end{equation*}
$$

### 3.3.3 Purely radial collisions

Since the KS naked singularity spacetimes demonstrate the "antigravity" or "repulsive gravity" effect that converts inward directed purely radial motion of uncharged test particles into an outward directed motion, similarly to an analogous effect occuring in the Kerr naked singularity spacetimes [128], we can consider a simple possibility to find the ultra-high-energy collisions by purely radial collision of two particles with energy $E=m$. The 4-velocities of the colliding particles read

$$
\begin{align*}
u_{1}^{\mu} & =\left(\frac{1}{f(r, \omega)}, \sqrt{1-f(r, \omega)}, 0,0\right)  \tag{3.27}\\
u_{2}^{\mu} & =\left(\frac{1}{f(r, \omega)},-\sqrt{1-f(r, \omega)}, 0,0\right) \tag{3.28}
\end{align*}
$$

The corresponding CM energy of the collision is given by formula

$$
\begin{equation*}
\frac{E_{C M}^{2}}{4 m^{2}}=\frac{1}{f(r, \omega)} \tag{3.29}
\end{equation*}
$$

The radial coordinate where we obtain the maximal CM energy of the collision in a given naked singularity spacetime is obtained when $f(r, \omega)$ has a minimum, i.e.

$$
\begin{equation*}
\frac{\mathrm{d} f(r, \omega)}{\mathrm{d} r}=0=\frac{6}{r^{2} A(r, \omega)}+2 r(1-A(r, \omega)) \omega . \tag{3.30}
\end{equation*}
$$

The behavior of the $E_{C M}^{2} / 4 m^{2}$ radial function, and of its maxima $E_{C M \text { max }}^{2} / 4 m^{2}$, is illustrated in Fig. 3.4. As magnitude of the parameter $\omega$ increases towards $\omega=0.5$, the radius of the maximum of CM energy approaches to $\mathrm{r}=1$. For $\omega \rightarrow 0.5$ the maximal value of $E_{C M}^{2} / 4 m^{2}$ diverges as $E_{C M}^{2} / 4 m^{2} \rightarrow 2 / 3 \delta$ as in the case of the collision in the direction perpendicular to the radial direction.

Figure 3.4.
The plots of the radial profiles of the CM energy (solid lines) and of its maximal value (dashed line) occurring due to the head-on collision of the radially moving particles. Each solid line is marked with the value of the corresponding parameter $\omega$.


Since the formula for the CM energy is the same for the collisions of particles with opposite angular momentum, and the radially colliding particles, we can immediately conclude that the ultra-high-energy radial collisions can occur in the radius $r=M$ in the field of the near-extreme KS naked singularities with the same dependence on the parameter $\delta$ for $\omega \rightarrow 1 / 2$. In both of these kinds of particle collisions, the centre-
of-mass reference system coincides with the static reference system at the position of the collision, as can be easily demonstrated following the arguments presented in [131]. Therefore, these cases can be used in a very simple way to estimate the efficiency of the ultra-high-energy collisions in relation to the static distant observers. We can construct the escape cones of the isotropically generated radiation, and calculate the frequency shift of the radiated high-energy photons at infinity, in an analogous way to the calculations in the Locally Non-Rotating Frames in the Kerr naked singularity spacetimes [131] that correspond to the static frames in the spherically symmetric spacetimes.

### 3.4. Motion of charged particles around naked singularities immersed in asymptotically uniform magnetic field

Finally we would like to study the collisions of charged particles moving in the combined gravitational and electromagnetic field assuming the gravitational field given by the KS naked singularity spacetimes, and the electromagnetic field to be an axially symmetric test magnetic field having asymptotically uniform form and strength $B>0$ [180]. The magnetic field is thus influenced by the spacetime, but its influence on the geometry is considered to be negligible. Motion of charged test particles is then governed by the general relativistic Lorentz equation. We shall consider collisions of charged particles following circular orbits in the equatorial plane of the combined gravitational and magnetic field, and electrically neutral particles incoming from rest state at infinity. Alternatively we consider collisions of oppositely charged particles moving along the circular orbits in opposite directions. Such collisions can in an appropriate way demonstrate the role of the magnetic field in the acceleration of the charged particles and the ultra-high energy collisions as discussed in an analogoues situation of a 5D black ring in [9].
3.4.1 Motion of charged particles and effective potential

The Hamiltonian of charged test particles moving in the combined gravitational and electromagnetic fields given by the metric $g_{\mu \nu}$ and potential of the electromagnetic field $A^{\mu 5}$ reads

$$
\begin{equation*}
H=\frac{1}{2} g^{\mu \nu}\left(\pi_{\mu}-e A_{\mu}\right)\left(\pi_{\nu}-e A_{\nu}\right) \tag{3.31}
\end{equation*}
$$

where $\pi_{\mu}$ are the components of generalized momentum. From the first set of Hamilton equations

$$
\begin{equation*}
\frac{\mathrm{d} x^{\mu}}{\mathrm{d} \lambda}=\frac{\partial H}{\partial \pi_{\mu}} \tag{3.32}
\end{equation*}
$$

one immediately obtains the relation

$$
\begin{equation*}
p^{\mu} \equiv \frac{\mathrm{d} x^{\mu}}{\mathrm{d} \lambda}=\pi^{\mu}-e A^{\mu} \tag{3.33}
\end{equation*}
$$

The metric and electromagnetic field of interest does not depend on coordinate time $t$ and on the azimuthal angle $-\phi$. From the second set of the Hamilton equations

$$
\begin{equation*}
\frac{\mathrm{d} \pi_{\mu}}{\mathrm{d} \lambda}=-\frac{\partial H}{\partial x^{\mu}}, \tag{3.34}
\end{equation*}
$$

we see two integrals of motion

$$
\begin{equation*}
\pi_{\phi}=L \text { and } \pi_{t}=-E . \tag{3.35}
\end{equation*}
$$

[^3]Introducing effective quantities, i.e., quantities expressed in unit mass

$$
\begin{equation*}
\mathcal{E}=\frac{E}{m}, \quad \mathcal{L}=\frac{L}{m}, \mathcal{B}=\frac{e B}{2 m^{2}}, \tag{3.36}
\end{equation*}
$$

the equations for the temporal, azimuthal and radial components of the 4 -momentum of the charged test particles then read

$$
\begin{align*}
p^{\phi} & =\frac{\mathcal{L}}{r^{2}}-\mathcal{B},  \tag{3.37}\\
p^{t} & =\frac{\mathcal{E}}{f}  \tag{3.38}\\
\left(p^{r}\right)^{2} & =\mathcal{E}^{2}-f\left[1+r^{2}\left(\frac{\mathcal{L}}{r^{2}}-\mathcal{B}\right)^{2}\right] . \tag{3.39}
\end{align*}
$$

The last equation determines the effective potential governing the radial motion, and the regions allowed for the particle motion

$$
\begin{equation*}
V_{e f f}=f(r)\left[1+r^{2}\left(\frac{\mathcal{L}}{r^{2}}-\mathcal{B}\right)^{2}\right] . \tag{3.40}
\end{equation*}
$$

3.4.2 Circular orbits of charged particles

For the KS naked singularities (black holes) immersed in an asymptotically uniform magnetic field, the circular orbits are allowed in the equatorial plane and are determined by the condition

$$
\begin{equation*}
\frac{\mathrm{d} V_{e f f}}{\mathrm{~d} r}=0 \tag{3.41}
\end{equation*}
$$

This equation implies a quadratic relation for the angular momentum of charged particles moving on a circular orbit at a given radius $r$. Therefore, we obtain two families of the circular orbits, with the angular momentum determined by the roots of the equation

$$
\begin{array}{r}
-\left(\frac{2}{r^{3}}+\omega A^{\prime}\right) L^{2}+4 B r \omega\left(-1+A+2 A^{\prime} / 2\right) L+r\left[2\left(B^{2}+\omega+2 B^{2} r^{2} \omega\right)\right. \\
\left.-2\left(\omega+2 B^{2} r^{2} \omega\right) A-r\left(1+B^{2} r^{2}\right) \omega A^{\prime}\right]=( \tag{3.42}
\end{array}
$$

The roots are

$$
\begin{equation*}
L_{c \pm}=\frac{-b \pm \sqrt{D}}{2 a} \tag{3.43}
\end{equation*}
$$

with

$$
\begin{align*}
a= & 1+r^{3} \omega A^{\prime} / 2,  \tag{3.44}\\
b= & 2 B r^{4} \omega(-1+A)+B r^{5} \omega A^{\prime},  \tag{3.45}\\
D= & r^{4}\left\{4\left(\omega+B\left(1+r^{2} \omega\right)^{2}\right)+\omega\left[4 B^{2} r^{4} \omega A^{2}-r A^{\prime}\left(2-2 r^{2} \omega+r^{3} \omega A\right.\right.\right. \\
& \left.\left.\left.-2 A\left(2+4 B^{2}\left(r^{2}+r^{4} \omega\right)+r^{3} \omega A^{\prime}\right)\right)\right]\right\} . \tag{3.46}
\end{align*}
$$

This equation implies directly the symmetry of the solutions related to the simultaneous transformations $L \rightarrow-L$ and $B \rightarrow-B$. Therefore, it is enough to study the case $\mathrm{B}>0$ only. The corresponding specific energy of the particles following the circular orbits of the two families given by Eq. (3.43) is given by

$$
\begin{equation*}
E_{c \pm}=V\left(r, L_{c \pm}\right) . \tag{3.47}
\end{equation*}
$$

The angular velocity relative to distant observers of the charged particle on the circular orbits reads

$$
\begin{equation*}
\Omega=\frac{\mathrm{d} \phi}{\mathrm{~d} t}=\frac{u^{\phi}}{u^{t}}=\frac{f}{r^{2}} \frac{L_{c \pm}}{E_{c \pm}}, \tag{3.48}
\end{equation*}
$$

with the angular momentum of the circular orbit $L$ given by formula (3.43).
The positions of stable (unstable) circular orbits of charged particle with angular momentum $L=L_{+}$and $L=L_{-}$in KS metric field and axisymmetric magnetic field B are represented in different regions in $(r, \omega)$ plane of Figs. 3.5-3.6.

Figure 3.5.
The shaded gray (light gray) regions represent $(r, \omega)$ positions of stable (unstable) circular orbits of charged particle with angular momentum $L=L_{+}$in KS metric field and axisymmetric magnetic field B.


Figure 3.6.
The shaded gray (light gray) regions represent ( $r, \omega$ ) positions of stable (unstable) circular orbits of charged particle with angular momentum $L=L_{-}$in KS metric field and axisymmetric magnetic field $B$.


### 3.5. Collisions of charged particles moving along circular orbits

We consider now two simple cases of the collisional processes of charged test particles that could well represent the role of the magnetic field added to the gravitational field of the KS naked singularities, as has been demonstrated in [9].

### 3.5.1. Collisions of particles at circular orbits with infalling neutral particles

The four-velocity of charged particles moving along circular orbits has the form

$$
\begin{equation*}
u_{1}^{\alpha}=\left(\gamma f^{-1 / 2}, 0,0, \gamma v / r\right), \tag{3.49}
\end{equation*}
$$

where $v$ is the velocity of the charged particle at the circular orbit at radius $r$ and $\gamma$ is the relativistic Lorentz factor, related to the static observers. Using the conditions $u^{\alpha} u_{\alpha}=-1$ and $d \phi / d \lambda=v \gamma / r$, one can easily find that

$$
\begin{equation*}
\gamma^{2}=1+\beta^{2} r^{2}, \quad v=\frac{\beta r}{\sqrt{1+\beta^{2} r^{2}}} \tag{3.50}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta=\frac{\mathcal{L}_{c}}{r^{2}}-\mathcal{B}, \tag{3.51}
\end{equation*}
$$

where the $L_{c}$ is the angular momentum of the charged particle at the circular orbit.
For the neutral particle incoming from infinity with specific energy $E$ and specific angular momentum $L$, the 4 -velocity is given by the relation

$$
\begin{equation*}
\left.u_{2}^{\alpha}=\left(\frac{\mathcal{E}}{f}, \sqrt{\mathcal{E}^{2}-f\left(1+\frac{\mathcal{L}_{2}^{2}}{r^{2}}\right)}, 0, \frac{\mathcal{L}_{2}}{r^{2}}\right)\right) \tag{3.52}
\end{equation*}
$$

Considering particles of identical rest mass $m_{1}=m_{2}=m$, and the neutral particle with $E=1$, we find the centre-of-mass energy to be given by

$$
\begin{align*}
\frac{E_{c m}^{2}}{2 m^{2}} & =1-g_{\alpha \beta} u_{1}^{\alpha} u_{2}^{\beta} \\
& =1-\mathcal{L}_{2} \beta+\sqrt{1+\beta^{2} r^{2}} f^{-1 / 2} \tag{3.53}
\end{align*}
$$

The results obtained for some characteristic values of the parameters $B$ and $\omega$ are illustrated in Fig. 3.7 demonstrating a critical role of the magnetic field enabling a significant enhancement of the acceleration of particles in the combined gravitational and electromagnetic fields of the KS naked singularities immersed in an uniform magnetic field.
3.5.2 Head-on collisions of oppositely charged particles at circular orbits

As a second example we discuss head-on collisions of the two oppositely charged particles having the same mass m and following the same circular orbit in opposite direction, i.e., having the same energy but opposite angular momentum. After the collision, the resulting 4-momentum of the system takes the form

$$
\begin{equation*}
P^{\alpha}=2 m \gamma f^{-1 / 2}(r \omega) \xi_{t}^{\alpha} . \tag{3.54}
\end{equation*}
$$

Figure 3.7. Centre of mass energy of head on collision of charged particle on circular orbit and neutral particle coming from infinity for the different values of parameter $B$ : the parameter $B=0.1,1.0,10.0$ and 100.0 (from top left to bottom right). The angular momentum of the neutral particle is taken to be $L_{n}=1$.


The centre-of-mass energy then takes a simple form

$$
\begin{equation*}
E_{C M}=2 m \gamma \tag{3.55}
\end{equation*}
$$

The results of calculations of the centre-of-mass energy are given for characteristic values of the the magnetic field intensity parameter B and the parameter $\omega$ in Fig. 3.8. We can see clearly that the role of the magnetic field can be very strong for large values of the parameter $B$.

Figure 3.8.
Centre of mass energy of headon collision of two charged particles on circular orbits. The parameter $B=0.1,1.0,10.0$ and 100.0 (from top left to bottom right).




3.6 Extreme rotating black hole in Horava-Lifshitz gravity

The four-dimensional metric of the spherical-symmetric spacetime written in the ADM formalism $[169,144]$ has the following form:

$$
\begin{equation*}
d s^{2}=-N^{2} c^{2} d t^{2}+g_{i j}\left(d x^{i}+N^{i} d t\right)\left(d x^{j}+N^{j} d t\right), \tag{3.56}
\end{equation*}
$$

where $N$ is the lapse function, $N^{i}$ is the shift vector to be defined.
The Ho rava-Lifshitz action describes a nonrelativistic renormalizable theory of gravitation and is given by (see for more details Ref. [160, 133, 144])

$$
\begin{align*}
S= & \int d t d x^{3} \sqrt{-g} N\left[\frac{2}{\kappa^{2}}\left(K_{i j} K^{i j}-\lambda_{g} K^{2}\right)\right. \\
& +\frac{\kappa^{2} \mu}{2 \nu_{g}^{2}} \epsilon^{i j k} R_{i l} \nabla_{j} R_{k}^{l}-\frac{\kappa^{2} \mu^{2}}{8} R_{i j} R^{i j}+\frac{\kappa^{2} \mu^{2}}{8\left(3 \lambda_{g}-1\right)} \\
& \left.\times\left(\frac{4 \lambda_{g}-1}{4} R^{2}-\Lambda_{W} R+3 \Lambda_{W}^{2}\right)-\frac{\kappa^{2}}{2 \nu_{g}^{4}} C_{i j} C^{i j}\right], \tag{3.57}
\end{align*}
$$

where $\kappa, \lambda_{g}, v_{g}, \mu$ and $\Lambda_{W}$ are constant parameters, the Cotton tensor is defined as

$$
\begin{equation*}
C^{i j}=\epsilon^{i k l} \nabla_{k}\left(R_{l}^{j}-\frac{1}{4} R \delta_{l}^{j}\right) \tag{3.58}
\end{equation*}
$$

$R_{i j k l}$ is the three-dimensional curvature tensor, and the extrinsic curvature $K_{i j}$ is defined as

$$
\begin{equation*}
K_{i j}=\frac{1}{2 N}\left(\dot{g}_{i j}-\nabla_{i} N_{j}-\nabla_{j} N_{i}\right), \tag{3.59}
\end{equation*}
$$

where dot denotes a derivative with respect to coordinate $t$.
If one considers up to second derivative terms in the action (3.57), one can find the known topological rotating solutions given by [181] for equations of motion in the Horava-Lifshitz gravity. Since we are considering matter coupled with the metric in a relativistic way, we can consider the metric in Boyer-Lindquist coordinates instead of its ADM form which is the solutions of Horava-Lifshitz gravity. In the Einstein's gravity this spacetime metric reads in Boyer-Lindquist type coordinates in the following form (see, e.g. [161]):

$$
\begin{align*}
d s^{2}= & -\frac{\Delta_{\mathrm{r}}}{\Sigma^{2} \rho^{2}}\left[d t-a \sin ^{2} \theta d \varphi\right]^{2}+\frac{\rho^{2}}{\Delta_{\mathrm{r}}} d r^{2} \\
& +\frac{\rho^{2}}{\Delta_{\theta}} d \theta^{2}+\frac{\Delta_{\theta} \sin ^{2} \theta}{\Sigma^{2} \rho^{2}}\left[a d t-\left(r^{2}+a^{2}\right) d \varphi\right]^{2}, \tag{3.60}
\end{align*}
$$

where the following notations

$$
\begin{aligned}
& \Delta_{\mathrm{r}}=\left(r^{2}+a^{2}\right)\left(1+\frac{r^{2}}{l^{2}}\right)-2 M r \\
& \Delta_{\theta}=1-\frac{a^{2}}{l^{2}} \cos ^{2} \theta \\
& \rho^{2}=r^{2}+a^{2} \cos ^{2} \theta \\
& \Sigma=1-\frac{a^{2}}{l^{2}}, l^{2}=-2 / \Lambda_{W}
\end{aligned}
$$

are introduced, M is the total mass of the central BH , a is the specific angular momentum of the BH . Note that metric (3.60) in ADM form can be written as [181]:

$$
\begin{align*}
d s^{2}= & -\frac{\rho^{2} \Delta_{\mathrm{r}} \Delta_{\theta}}{\Sigma^{2} \Xi^{2}} d t^{2}+\frac{\rho^{2}}{\Delta_{\mathrm{r}}} d r^{2}+\frac{\rho^{2}}{\Delta_{\theta}} d \theta^{2} \\
& +\frac{\Xi^{2} \sin ^{2} \theta}{\Sigma^{2} \rho^{2}}[d \varphi-\varpi d t]^{2}, \tag{3.61}
\end{align*}
$$

where

$$
\begin{aligned}
& \Xi^{2}=\left(r^{2}+a^{2}\right)^{2} \Delta_{\theta}-a^{2} \Delta_{\mathrm{r}} \sin ^{2} \theta \\
& \varpi=-\frac{a}{\Xi^{2}}\left[-\left(r^{2}+a^{2}\right) \Delta_{\theta}+\Delta_{\mathrm{r}}\right]
\end{aligned}
$$

The spacetime (3.60) has a horizon where the four velocity of a corotating observer tends to zero, or the surface $r=$ const becomes null. Thus we have:

$$
\begin{equation*}
r_{+} \simeq(1-3 \delta)\left[M+\sqrt{M^{2}-a^{2}(1+3 \delta)}\right], \tag{3.62}
\end{equation*}
$$

where we have introduced small dimensionless parameter $\delta=a^{2} / l^{2} \ll 1$.
The static limit is defined where the time-translation Killing vector $\xi_{(t)}^{\alpha}(t)$ becomes null (i.e. $g_{00}=0$ ), so the static limit of the BH can be described as

$$
\begin{align*}
r_{\mathrm{st}} & \simeq(1-3 \delta)\{M \\
& \left.+\sqrt{M^{2}-a^{2}\left(1+\delta \sin ^{2} \theta\right)(1+3 \delta) \cos ^{2} \theta}\right\} \tag{3.63}
\end{align*}
$$

In the recent paper [181] authors provide the ADM and Boyer-Lindquist forms of the spacetime metric in Ho rava-Lifshitz gravity both. From the expression (3.61) one can easily see that the results for the radii of event horizon and static limit will be identical in the the ADM and Boyer-Lindquist ones (For more details about these forms of spacetime metric presentation in Ho rava-Lifshitz gravity we refer to the papers [161] and [181].).

Considering only the outer horizon, $r_{+}$and static limit, $r_{s t}$, it can be verified that the static limit always lies outside the horizon. The region between the two is called the ergosphere, where timelike geodesics cannot remain static but can remain stationary due to corotation with the BH with the specific frame dragging velocity at the given location in the ergosphere. This is the region of spacetime where timelike particles with negative angular momentum relative to the BH can have negative energy relative to the infinity. Then, energy could be extracted from the hole by the well-known Penrose process [182].

In Fig. 3.9 the dependence of the shape of the ergosphere from the small dimensionless parameter $\delta$ is shown. From the figure one can see, that the relative shape of the ergosphere becomes bigger with increasing the module of the parameter $\delta$. Although in the polar region there is no ergoregion in the presence of the nonvanishing $\delta$ parameter, where Penrose process can be realized, near to polar zone ergoregion becomes more bigger than that in the Kerr spacetime. It may increase the efficiency of the Penrose process.

Figure 3.9.
The dependence of the shape of the ergosphere from the small dimensionless parameter $\delta$ : a) $\delta=0$, b) $\delta=0.001$ c) $\delta=0.01$, d) $\delta=0.1$.


c)

d)

### 3.7. Energy extraction of black hole through Penrose process

Due to the existence of an ergosphere around the BH , it is possible to extract energy from BH by means of the Penrose process. Inside the ergosphere, it is possible to have a timelike or null trajectory with the negative total energy. As a result, one can envision a particle falling from infinity into the ergosphere and splitting into two fragments, one of which attains negative energy relative to the infinity and falls into the hole at the pole, while the other fragment would come out by conservation of energy with the energy being greater than that of the original incident particle. This is how the energy could be extracted from the hole by axial accretion of particles with the nonvanishing momentum and $\delta$ parameter.

Consider the equation of motion of such negative energy particle at the equatorial plane $(\theta=\pi / 2, \dot{\theta}=0)$. Using the Hamilton-Jacobi formalism the energy $E$ and angular momentum $L$ of the particle are given as (see, e.g [183])

$$
\begin{align*}
-\tilde{E} & =-\frac{E}{m}=\left[-\frac{1}{\Sigma^{2}}\left(1-\frac{2 M}{r}-\frac{r^{2}+a^{2}}{l^{2}}\right)\right] \dot{t}+\frac{a}{\Sigma^{2}}\left(\frac{r^{2}+a^{2}}{l^{2}}-\frac{2 M}{r}\right) \dot{\varphi}  \tag{3.64}\\
\tilde{L} & =\frac{L}{m}=\frac{1}{\Sigma^{2}}\left(r^{2}+a^{2} \frac{l^{2}-r^{2}-a^{2}}{l^{2}}+\frac{2 M a^{2}}{r}\right) \dot{\varphi}+\frac{a}{\Sigma^{2}}\left(\frac{r^{2}+a^{2}}{l^{2}}-\frac{2 M}{r}(3)\right. \tag{3.65}
\end{align*}
$$

From the equations (3.64)-(3.65) one can easily obtain the equation of motion as:

$$
\begin{equation*}
\alpha E^{2}+\beta E+\gamma+\frac{\rho^{2}}{\Delta_{\mathrm{r}}} \dot{p}^{\mathrm{r}^{2}}+m^{2}=0 \tag{3.66}
\end{equation*}
$$

where we have introduced the following notations:

$$
\begin{align*}
& \alpha=\frac{1}{\Sigma^{2}}\left(r^{2}+a^{2}-a^{2} \frac{r^{2}+a^{2}}{l^{2}}+a^{2} \frac{2 M}{r}\right) \Gamma^{-1}  \tag{3.67}\\
& \beta=\frac{2 a L}{\Sigma^{2}}\left(\frac{r^{2}+a^{2}}{l^{2}}-\frac{2 M}{r}\right) \Gamma^{-1}  \tag{3.68}\\
& \gamma=-\frac{L^{2}}{\Sigma^{2}}\left(1-\frac{2 M}{r}+\frac{r^{2}+a^{2}}{l^{2}}\right) \Gamma^{-1} \tag{3.69}
\end{align*}
$$

and

$$
\begin{align*}
\Gamma= & -\frac{1}{\Sigma^{4}}\left(1-\frac{2 M}{r}+\frac{r^{2}+a^{2}}{l^{2}}\right)\left(r^{2}+a^{2} \frac{l^{2}-a^{2}-r^{2}}{l^{2}}+a^{2} \frac{2 M}{r}\right) \\
& -\frac{a^{2}}{\Sigma^{4}}\left(\frac{r^{2}+a^{2}}{l^{2}}-\frac{2 M}{r}\right)^{2} . \tag{3.71}
\end{align*}
$$

From the equations (3.64), (3.65) (3.66) one can easily obtain the equations of motion in the following form:

$$
\begin{align*}
\frac{d t}{d s} & =\frac{\Sigma^{2}}{r^{2} \Delta_{\mathrm{r}}}\left\{\left[\left(r^{2}+a^{2}\right)^{2}-\Delta_{\mathrm{r}} a^{2}\right] E+\left(\Delta_{\mathrm{r}}-r^{2}-a^{2}\right) a L\right\}  \tag{3.72}\\
\frac{d \varphi}{d s} & =\frac{\Sigma^{2}}{r^{2} \Delta_{\mathrm{r}}}\left\{\left(\Delta_{\mathrm{r}}-a^{2}\right) L+\left(r^{2}+a^{2}-\Delta_{\mathrm{r}}\right) E\right\}  \tag{3.73}\\
\left(\frac{d r}{d s}\right)^{2} & =E^{2}-V_{\mathrm{eff}} \\
V_{\mathrm{eff}} & =\left(1+\frac{\Delta_{\mathrm{r}} \alpha}{\rho^{2}}\right) E^{2}+\frac{\Delta_{\mathrm{r}}}{\rho^{2}}(\beta E+\gamma+1) . \tag{3.74}
\end{align*}
$$

In the Fig. 3.10 the radial dependence of the effective potential of radial motion of the massive test particle has been shown for the different values of the parameter $\delta$. Here for the energy and momenta of the particle the following values are taken: $E / m=$
$0.9, L / m M=6.2$. The presence of the parameter $\delta$ slightly shifts the shape of the effective potential up.

Figure 3.10.
The radial dependence of the effective potential of radial motion of the massive test particle for the different values of the dimensionless parameter $\delta$.


When the one of two produced particles falls into the central BH, the mass of the BH will change by $\Delta M=E$. The change in mass can be made as large as one pleases by increasing the mass m of the infalling particle. However, there is a lower limit on $\Delta M$ which could be added to the BH corresponding to $m=0$ and $\dot{p}_{r}=0$ [183]. Evaluating all of the required quantities at the horizon $r_{+}$, one can easily get the limit for the change in BH mass as

$$
\begin{equation*}
E_{\min }=L \frac{\delta\left(a^{2}+r_{+}^{2}\right) / a-2 M a / r_{+}}{r_{+}^{2}+a^{2}-a^{2} \delta-r^{2} \delta+a^{2} 2 M / r_{+}} . \tag{3.75}
\end{equation*}
$$

From the expression (3.75) one may conclude that Penrose process can be realized if the condition $\delta<2 M a^{2} / r_{+}\left(a^{2}+r_{+}^{2}\right)$ will be satisfied. Since current astrophysical data indicate that parameter is much less than 1 , one may conclude that Penrose process is more realistic process among the energy extraction mechanisms from BH in Horava-Lifshitz scenario. However, it should be mentioned that in the early Universe when the module of the cosmological constant played important role, energy extraction from the rotating BH could be impossible in Horava-Lifshitz scenario due to the positivity of the sign of $E_{\text {min }}$. This limitation for Penrose process does not exist in the standard theory gravity and appears in the modified theory gravity as HoravaLifshitz one.
3.8. Particle acceleration near the rotating black hole in Horava-Lifshitz gravity

Let us find the energy Ecm in the center of mass of system of two colliding particles with energy at infinity $E_{1}$ and $E_{2}$ in the gravitational field described by spacetime metric (3.60). It can be obtained from

$$
\begin{equation*}
\left(\frac{1}{\sqrt{-g_{00}}} E_{\mathrm{cm}}, 0,0,0\right)=m_{1} v_{(1)}^{\alpha}+m_{2} v_{(2)}^{\alpha}, \tag{3.76}
\end{equation*}
$$

where $v_{(1)}^{\mu}$ and $v_{(2)}^{\mu}$ are the 4 -velocities of the particles, properly normalized by $g_{\mu \nu} v^{\mu} v^{v}=-1$ and $m_{1}, m_{2}$ are rest masses of the particles. We will consider two particles with equal mass ( $m_{1}=m_{2}=m_{0}$ ) which has the energy at infinity $E_{1}=E_{2} \cong$ 1. Thus we have

$$
\begin{equation*}
E_{\mathrm{cm}}=m_{0} \sqrt{2} \sqrt{1-g_{\alpha \beta} v_{(1)}^{\alpha} v_{(2)}^{\beta}} . \tag{3.77}
\end{equation*}
$$

Now using the equations (3.72)-(3.74) one can obtain expression for the energy of colliding particles near the Ho rava-Lifshitz black hole as:

$$
\begin{align*}
E_{\mathrm{c.m.} .}^{2}= & \frac{2 m_{0}^{2} \Sigma^{2}}{r \Delta_{r}}\left\{\frac{\delta r^{5}}{a^{2}}+2 r^{2}[r(1+2 \delta)-1-4 \delta]-\left[2 a-\delta r a\left(1+r^{2}\right)\right]\left(l_{1}+l_{2}\right)\right. \\
& -\sqrt{2\left(a-l_{1}\right)^{2}-\left(a^{2} \delta-2 a \delta l_{1}+l_{1}^{2}+l_{1}^{2} \delta+2 r\right) r-2 \delta+2 \frac{\delta l_{1}}{a}-\frac{l_{1}^{2}+r^{2}}{a^{2}} \delta r^{3}} \\
& \left.\times \sqrt{2\left(a-l_{2}\right)^{2}-\left(a^{2} \delta-2 a \delta l_{2}+l_{2}^{2}+l_{2}^{2} \delta+2 r\right) r-2 \delta+2 \frac{\delta l_{2}}{a}-\frac{l_{2}^{2}+r^{2}}{a^{2}} \delta r^{3}}\right\} \\
& +l_{1} l_{2}\left[2-r+\delta r\left(1+\frac{r^{2}}{a^{2}}\right)\right]+2 a^{2}[1+(1+1.5 \delta) r] \tag{3.78}
\end{align*}
$$

In the Fig. 3.11 the radial dependence of the center mass energy of two particles for the different values of the dimensionless parameter $\delta$ has been shown. From the Fig. 3.11 one can easily see that in the Horava-Lifshitz gravity the particle can essentially accelerate near the horizon but not to arbitrary high energies. With increasing the parameter $\delta$ the maximal value of the center of mass energy is decreasing.

Banados, Silk and West [84] have shown that the total energy of two colliding test particles has no upper limit in their center of mass frame in the neighborhood of an extreme Kerr black hole, even if these particles were at rest at infinity in the infinite past. On the contrary, we show here that the energy of two colliding particles in the center of mass frame observed from the infinity has an upper limit in the HoravaLifshitz gravity.

Figure 3.11.
The radial dependence of the center of mass energy of two infalling particles for the different values of the parameter $\delta$ in the case of the extreme black hole $(a=M)$.


### 3.9. Conclusions

We have demonstrated that at the KS naked singularity spacetimes, at least three kinds of collisions of particles falling freely from infinity could lead to ultrahigh-energy observed in the CM system. Two types of the collisions can occur quite naturally in the field of near-extreme KS naked singularities at the specific radius $r=M$, in close analogy to the collisions in the Kerr naked singularity spacetimes [131]. There are only small quantitative differences between the KS and Kerr cases for the purely radial collisions. At this point the notion of the naked singularity is decisive qualitatively. Moreover, we can see that the local efficiency of the collisions of the particles in purely radial direction is identical to the efficiency obtained in the case of purely axial motion of the particles if they collide at the turning point of their radial motion.

However, the efficiency of the ultra-high-energy processes, occurring at the specific radius $r=M$, and related to the distant observers, differ substantially in the KS and Kerr naked singularity spacetimes. In the near-extreme KS case, the efficiency is strongly restricted for both the escaping and the frequency shift, and is similar to the black hole case. In fact, the gravitational redshift cancels the energy excess obtained due to the collisional process and only the covariant energy of the colliding particles (rest energy for particles freely falling from rest at infinity) is relevant for energy observed at infinity. Such a system works as a standard accelerator of particles.

On the other hand, in the near-extreme Kerr case, both these phenomena can be enhanced relatively very strongly, if the collisions occurs near the equatorial plane of the Kerr geometry where the rotational effects of the Kerr background corresponding to efficient Penrose processes are strongest [131]. We can conclude that for the high efficiency of the ultra-high-energy processes relative to distant observers, both the nonexistence of the horizon, and the strong rotational effects are necessary, if we do not consider relevance of electromagnetic phenomena.

We have to stress that significant magnification of the efficiency of the ultrahigh energy collisions is possible due to additional electromagnetic phenomena influencing collisions of charged particles, as demonstrated here for simple situations related to the circular motion of charged particles orbiting near the KS naked singularities immersed in an asymptotically uniform magnetic field. Of course, this phenomenon occurs also in the collisions near black holes [1, 9, 147, 148].

We have studied the energetics of the rotating black hole in Horava-Lifshitz gravity. Fisrt, we considered the energy extraction mechanism via the Penrose process and found exact expression for limit for the change in BH mass (3.75) and concluded that Penrose process can be realized if the condition $\delta<2 M / r_{+}$will be satisfied. Since the parameter $\delta$ is much less than 1 , it is easy to conclude that energy extraction through Penrose process is more realistic process among the energy extraction mechanisms
from BH in Horava-Lifshitz scenario. However, it should be mentioned that in the early Universe when the module of the cosmological constant played important role, energy extraction from the rotating BH could be impossible in Horava-Lifshitz scenario due to the positivity of the sign of $E_{\text {min }}$. This limitation for Penrose process does not exist in the standard theory gravity and appears in the modified theory gravity as HoravaLifshitz one.

In the paper [84] authors underlined, that rotating black hole can accelerate the particles falling to the central black hole to arbitrary high energies. Because of some mechanisms such as astrophysical limitations on the maximal spin, back-reaction effects, and sensitivity to the initial conditions, there appears some upper limit for the center of mass energy of the infalling particles. One of the mechanisms offered in this work is appearing due to the Horava-Lifshitz gravity correction, which prevents particle from the infinite acceleration.

# CHAPTER IV. ELECTRODYNAMICS AND SPIN DOWN OF MAGNETIZED NEUTRON STARS 

### 4.1. Introduction

The study of electromagnetic fields of magnetized compact objects is an important task for several reasons. First, we obtain information about such stars through their observable characteristics, which are closely connected with electromagnetic fields inside and outside the relativistic stars. Magnetic fields play an important role in the life history of majority astrophysical objects especially of compact relativistic stars which possess surface magnetic fields of $10^{12} \mathrm{G}$ and $\sim 10^{14} \mathrm{G}$ in the exceptional cases for magnetars [184, 185]. The strength of compact star's magnetic field is one of the main quantities determining their observability, for example as pulsars through the magneto-dipolar radiation. Electromagnetic waves radiated from the star determine energy losses from the star and therefore may be related with such observable parameters as period of pulsar and it's time derivative. The second reason is that we may test various theories of gravitation through the study of compact objects for which general relativistic effects are especially strong. Considering different matter for the stellar structure one may investigate the effect of the different phenomena on evolution and behavior of stellar interior and exterior magnetic fields. Then these models can be checked through comparison of theoretical results with the observational data. The third reason may be seen in the influence of stellar magnetic and electric field on the different physical phenomena around the star, such as gravitational lensing and motion of test particles.

Among astrophysical objects which can be useful in investigating the physics under the extreme conditions particular place belongs to radio pulsars. According to the magnetospheric models radio pulsars are rotating highly magnetized neutron stars,
producing radio emission above the small area of its surface called polar cap. [186] proved that such a rotating highly magnetized star cannot be surrounded by vacuum due to generation of strong electric field pulling out charged particles from the surface of the star. They proposed first model of the pulsar magnetosphere containing two distinct regions: the region of closed magnetic field lines, where plasma corotates with the star as a solid body, and the region of open magnetic field lines, where radial electric field is not completely screened with plasma particles and plasma may leave the neutron star along magnetic field lines. Radio emission is generated due to continuous cascade generation of electron-positron pairs in the magnetosphere above the polar cap. Thorough research on structure and physical processes in pulsar magnetosphere can be found in works of [186, 187, 188, 189, 190], [191]. Although a self-consistent pulsar magnetosphere theory is yet to be developed, the analysis of plasma properties in the pulsar magnetosphere based on the above-mentioned papers provides firm ground for the construction of such a model.

It was shown by a number of authors that effects of general relativity play very important role in physics of pulsars. The effect of general relativistic frame dragging effect in the plasma magnetosphere was investigated in [192, 193, 191, 127 194, 195], and many others, and proved to be crucial for the conditions of particle acceleration in the magnetosphere and, therefore, for generation of radio emission. The effect of the stellar oscillations on plasma magnetosphere in general relativity is recently discussed in [196, 197, 198, 199]. From the above mentioned papers it is seen that the general relativistic effects in the plasma magnetosphere of pulsars are not negligible and should be carefully considered.

The majority of neutron stars are known to have large angular velocities, and in the case of radio pulsars one can directly measure their speed of rotation. It is also observed that, on average, their rotation tends to slow down with time, a phenomenon that is explained by emission of electromagnetic waves or, in some conditions, by the
emission of gravitational waves or other processes. This should be the case during most of the life of the neutron star when it is observed as pulsar. Since 1967 [200] pulsars play a role of relativistic astrophysical laboratory where the fast rotation, the extreme density of matter and the high magnetic field are realized.

Neutron stars provide a natural laboratory to study extremely dense matter. In the interiors of such stars, the density can reach up to several times the nuclear saturation density $n_{0} \cong 0.16 \mathrm{fm}^{-3}$. At such high densities quarks could be squeezed out of nucleons to form quark matter. The true ground state of dense quark matter at high densities and low temperatures remains an open problem due to the difficulty of solving nonperturbative quantum chromodynamics (QCD). It has been suggested that strange quark matter that consists of comparable numbers of $u$, $d$, and $s$ quarks may be the stable ground state of normal quark matter. This has led to the conjecture that the family of compact stars may have members consisting entirely of quark matter (so-called strange stars) and/or members featuring quark cores surrounded by a hadronic shell (hybrid stars). The physics of strange matter is reviewed for example by [201, 202, 203, 204, 205].

The possibility of existence of stable self-bound strange matter could have important consequences for neutron stars: some compact, dense stars could be strange stars. Discovery of a strange star in the universe would be a confirmation of the validity of our present theory of the structure of matter. In view of this, we should look for the signatures of strange stars. Univocal and detectable signature of strange star would be a key to its possible detection. Low mass strange stars are much smaller than neutron stars. There is no lower limit for strange star mass.

Observationally, it is very challenging to distinguish the various types of compact objects, such as the strange stars, hybrid stars, and ordinary neutron stars. Newly born strange stars are much more powerful emitters of neutrinos than neutron stars. Their early cooling behavior is dominated by neutrino emission which is a useful probe of the
internal composition of compact stars. Thus, cooling simulations provide an effective test of the nature of compact stars. However, many theoretical uncertainties and the current amount of data on the surface temperatures of neutron stars leave sufficient room for speculations. However, this property is also characteristic of the neutron star with a large quark core.

Another useful avenue for testing the internal structure and composition of compact stars is astroseismology, i.e., the study of the phenomena related to stellar vibrations. Pulsations of newly born strange star are damped in a fraction of second: after this time copious neutrino flux from them should not show pulsating features. Unfortunately, the same would be true for the neutrino emission from a neutron star with a large quark core.

Photon cooling of bare strange star could be significantly different from that of neutron stars. If quark surface is not an extremely poor emitter of photons, then absence of insulating crust could lead to a relatively fast photon cooling. In principle, a well established upper limit of surface temperature of neutron starlike object of known age, which is well below the estimates for an object with crust, could be a signature of a bare strange star. A neutron star-like object with crust which is 10 y old cannot have a surface temperature lower than $10^{6} \mathrm{~K}$. Unfortunately, surface (black-body) emission flux decreases as a fourth power of temperature and at the present day with X-ray satellite detector it is very difficult to detect an object of 10 km radius at 100 pc (typical distance to the nearest observed point X-ray sources) if its surface black body temperature is quite low. However the recent observation of radio-quiet, X-ray bright, central compact objects (CCOs) shows that the spectra of these objects can very well be described by a one- or two-component blackbody model, which would indicate unusually small radii ( $\sim 5 \mathrm{~km}$ ) for these objects [206]. Such small radii can only be explained in terms of selfbound stellar objects like strange quark matter star.

The search for the detectable signatures of strange stars should be continued. After all, chances of producing strange matter in our laboratories are negligibly small compared to those for its creation during $10^{10}$ years in the immense cosmic laboratory of the universe.

The chapter is based on the following papers $[2,4,5]$ of the author. In this chapter we will be concerned with the possibility to distinguish neutron star from the strange star from the spin down of pulsar. We compare the spindown features of neutron star and strange star and discuss if astrophysical observations could be useful to prove the validity of the strange matter hypothesis. then we plan to extend this study to the magnetospheric case. We plan to compare the spin down features of neutron star and strange star in the presence of the plasma magnetosphere and discuss if astrophysical observations could be useful to prove the validity of the strange matter hypothesis.
4.2 Vacuum electrodynamics and spin down of the strange and neutron stars

If strange stars are born in some supernova explosions then because of enormous electric conductivity of strange matter they should possess huge frozen-in magnetic field [207]. In this respect strange stars could be used as models of pulsars. The calculation of electric conductivity $\sigma$ of strange matter has been done by [204]. Charge transport in strange matter is dominated by quarks. The value of conductivity $\sigma$ is determined by the color-screened QCD interaction $\sigma=10^{29} T_{10}^{-2} \mathrm{~s}^{-1}$ and it is only several times larger than electric conductivity of normal neutron star matter of the same density and temperature. In the case f neutron star matter the charge carriers are electrons.

The magnetic field of strange stars will decay due to ohmic dissipation of currents strange matter. For the dipole magnetic field the decay time is [208]

$$
\begin{equation*}
\tau_{D} \cong \frac{4 \sigma R^{2}}{\pi c^{2}} \tag{4.1}
\end{equation*}
$$

and for $T<10^{9} \mathrm{~K}$ and $R=10 \mathrm{~km}$ one can get $\tau_{D}>4 \cdot 10^{11} \mathrm{yr}$. The ohmic dissipation of magnetic field in a strange star is thus negligible.

In this section we plan to study the spin-down of a rotating strange star due to magnetodipolar electromagnetic emission. Assume that the oblique rotating magnetized star is observed as radio pulsar through magnetic dipole radiation. Then the luminosity of the relativistic star in the case of a purely dipolar radiation, and the power radiated in the form of dipolar electromagnetic radiation, is given as [111]

$$
\begin{equation*}
L_{e m}=\frac{\Omega_{R}^{4} R^{6} \widetilde{B}_{0}^{2}}{6 c^{3}} \sin ^{2} \chi \tag{4.2}
\end{equation*}
$$

where tilde denotes the general relativistic value of the corresponding quantity, subscript $R$ denotes the value of the corresponding quantity at $r=R$ and $\chi$ is the inclination angle between magnetic and rotational axes. Here we will use the spacetime of slowly rotating relativistic star which in a coordinate system $(t, r, \theta ; \phi)$ has the following form:

$$
\begin{equation*}
d s^{2}=-e^{2 \Phi(r)} d t^{2}+e^{2 \Lambda(r)} d r^{2}-2 \omega(r) r^{2} \sin ^{2} \theta d t d \varphi+r^{2} d \theta^{2}+r^{2} \sin ^{2} \theta d \varphi^{2}( \tag{4.3}
\end{equation*}
$$

where metric functions $\Phi$ and $\Lambda$ are completely known for the outside of the star and given as:

$$
\begin{equation*}
e^{2 \Phi}=\left(1-\frac{2 M}{r}\right)=e^{-2 \Lambda}, \tag{4.4}
\end{equation*}
$$

$\omega=2 J / r^{3}, J$ is the total angular momentum of the star with total mass $M$ and moment of inertia $I$.

For the interior of the star the metric functions strongly depend on equation of state and was widely discussed in the literature [209, 210, 211]. In particular, the function $\Lambda(r)$ is related to mass function $m(r)$ :

$$
\begin{equation*}
e^{2 \Lambda}=\left(1-\frac{2 m}{r}\right)^{-1} \tag{4.5}
\end{equation*}
$$

Function $\Phi(r)$ can be determined from evaluating the following integral [209]:

$$
\begin{equation*}
\Phi(r)=\frac{1}{2} \ln \left(1-\frac{2 M}{r}\right)-\int_{r}^{R} \frac{m(x)+4 \pi x^{3} p(x)}{x^{2}(1-2 m(x) / x)} d x \tag{4.6}
\end{equation*}
$$

where $p(x)$ is the pressure profile over the parameter $0 \leq x \leq R$.
When compared with the equivalent Newtonian expression for the rate of electromagnetic energy loss through dipolar radiation [208, 212],

$$
\begin{equation*}
\left(L_{\text {em }}\right)_{\text {Newt. }}=\frac{\Omega^{4} R^{6} B_{0}^{2}}{6 c^{3}} \sin ^{2} \chi, \tag{4.7}
\end{equation*}
$$

it is easy to realize that the general relativistic corrections emerging in expression (4.2) are due partly to the magnetic field amplification at the stellar surface

$$
\begin{equation*}
\frac{\widetilde{B}_{0}}{B_{0}}=\frac{\widetilde{B}_{0} R^{3}}{2 \mu}=f_{R}=-\frac{3 R^{3}}{8 M^{3}}\left[\ln N_{R}^{2}+\frac{2 M}{R}\left(1+\frac{M}{R}\right)\right], \tag{4.8}
\end{equation*}
$$

and partly to the increase in the effective rotational angular velocity produced by the gravitational redshift

$$
\begin{equation*}
\Omega(r)=\Omega_{R} \frac{N_{R}}{N}=\Omega_{R} \sqrt{\left(\frac{R-2 M}{r-2 M}\right) \frac{r}{R}}, \tag{4.9}
\end{equation*}
$$

i.e. $\Omega=\Omega_{R} N_{R}=\Omega_{R} \sqrt{1-2 M / R}$ when $r \rightarrow \infty$, where $N=(1-2 M / r)^{1 / 2}$ is the lapse function.

Expression (4.2) could be used to investigate the rotational evolution of magnetized neutron stars with predominant dipolar magnetic field anchored in the crust which converts its rotational energy into electromagnetic radiation. First detailed investigation of general relativistic effects for Schwarzschild stars has been performed by [211], who have paid special attention to the general relativistic corrections that needed to be included for a correct modeling of the thermal evolution but also of the magnetic and rotational evolution.

Therefore, the presence of a curved spacetime has the effect of increasing the rate of energy loss through dipolar electromagnetic radiation for the strange star with comparison to that for neutron star by an amount which can be easily estimated to be

$$
\begin{equation*}
\frac{\left(L_{e m}\right)_{\mathrm{ss}}}{\left(L_{e m}\right)_{\mathrm{NS}}}=\left(\frac{f_{R}}{N_{R}^{2}}\right)_{\mathrm{ss}}^{2} /\left(\frac{f_{R}}{N_{R}^{2}}\right)_{\mathrm{NS}}^{2} . \tag{4.10}
\end{equation*}
$$

The expression for the energy loss (4.2) can also be used to determine the spinevolution of a pulsar that converts its rotational energy into electromagnetic radiation. Following the simple arguments proposed more than thirty years ago [212, 213, 214],
it is possible to relate the electromagnetic energy loss $L_{e m}$ directly to the loss of rotational kinetic energy $E_{\text {rot }}$ defined as

$$
\begin{equation*}
E_{\mathrm{rot}} \equiv \frac{1}{2} \int d^{3} \mathrm{x} \sqrt{\gamma} e^{-\Phi(r)} \rho\left(\delta v^{\hat{\varphi}}\right)^{2}, \tag{4.11}
\end{equation*}
$$

where $\rho$ is the stellar energy density and factor $\gamma$ is defined as follow:

$$
\gamma=\left[-g_{00}\left(1+g_{i k} \frac{\delta v^{i} \delta v^{k}}{g_{00}}\right)\right]^{-1 / 2} \simeq e^{-\Phi}
$$

$\delta v^{i}$ is the three velocity of conducting medium defined in [111].
Authors of the paper [209] have shown that having generated an equation of state it is easy to find pressure $p(r)$ and density $\rho(r)$ profile. In most physical models considered in the literature, for example in the paper [209] the ratio $p / \rho$ in the crust lies in the interval $(4 \div 5) \cdot 10^{-3}$. As one moves towards the center of the star this ratio could increase to as large as $10 \%$. However the moment of the inertia is very sensitive to the matter density/pressure in the crust of the star and one can assume that the condition $p / \rho \ll 1$ will approximately satisfy, where $p$ is the pressure of the stellar matter. This is suitable in the most physical situations and one can introduce the general relativistic moment of the inertia of the star as [111, 215, 209, 216, 217]:

$$
\begin{equation*}
\widetilde{I} \equiv \int d^{3} \mathbf{x} \sqrt{\gamma} e^{-\Phi(r)} \rho r^{2} \sin ^{2} \theta \tag{4.12}
\end{equation*}
$$

whose Newtonian limit gives the well-known expression $I \equiv(\tilde{I})_{\text {Newt }}=\frac{2}{5} M R^{2}$, the energy budget is then readily written as

$$
\begin{equation*}
\dot{E}_{\mathrm{rot}} \equiv \frac{d}{d t}\left(\frac{1}{2} \widetilde{I} \Omega^{2}\right)=-L_{e m} \tag{4.13}
\end{equation*}
$$

Of course, in enforcing the balance (4.13) we are implicitly assuming all the other losses of energy (e.g. those to gravitational waves) to be negligible. This can be a reasonable approximation except during the initial stages of the pulsar's life, during which the energy losses due to emission of gravitational radiation will dominate because of the steeper dependence on the angular velocity.

Expression (4.13) can also be written in a more useful form in terms of the pulsar's most important observables: the period $P$ and its time derivative $\dot{P} \equiv d P / d t$. In this case, in fact, using expression (4.2) and (4.13), it is not difficult to show that

$$
\begin{equation*}
(P \dot{P})_{\mathrm{ss}}=\left(\frac{f_{R}^{2}}{N_{R}^{4}}\right)_{\mathrm{sS}}\left(\frac{f_{R}^{2}}{N_{R}^{4}}\right)_{\mathrm{NS}}^{-1} \frac{\widetilde{I}_{\mathrm{NS}}}{\widetilde{I}_{\mathrm{SS}}}(P \dot{P})_{\mathrm{NS}} . \tag{4.14}
\end{equation*}
$$

Also in this case it is not difficult to realize that general relativistic corrections will be introduced through the amplification of the magnetic field and of the stellar angular velocity, as well as of the stellar moment of inertia.

Considering slowly rotating magnetized neutron star one can see that the general relativistic corrections emerging in expression (4.2) will be partly due to the magnetic field amplification at the stellar surface and partly to the increase in the effective rotational angular velocity produced by the gravitational redshift.

General-relativistic treatment for the structure of external and internal stellar magnetic fields including numerical results has shown that the magnetic field is amplified by the monopolar part of gravitational field depending on the compactness of the relativistic star. Thus for a given compact star, the effects of general relativity on
electromagnetic luminosity can be characterized only by the single compactness parameter $\mathrm{M}=\mathrm{R}$ which is different for the neutron and strange star.

Let us mention the so called canonical neutron star model used by many authors. This artificial model does not imply any specific EOS, but just assumes the typical values of $M$ and $R: M=1.4 M_{\text {Sun }}, R=10 \mathrm{~km}$. Using the data for the mass, the radius, the moment of inertia of neutron stars and strange stars from the recent paper [218] we have calculated the ratio of spin down of neutron star to one of the strange star on the base formula (4.14) for the compact stars of the different masses. Results are summarized in the Table 4.1 from where one can see that the strange star is spinning down approximately 5 times faster that the neutron star.

Table 4.1.
The dependence of the ratio $(P \dot{P})_{S S} /(P \dot{P})_{N S}$ from the different parameters of the compact object: mass (in units of solar mass), radii and moment of inertia of the Strange $\left(R_{S S}, I_{S S}\right)$ and Neutron stars $\left(R_{N S}, I_{N S}\right)$. Data for strange and neutron stars are obtained from the paper [218].

| $(P \dot{P})_{S S} /(P \dot{P})_{N S}$ | 4,34463 | 4.53723 | 5.1094 | 6.16863 |
| :--- | :---: | :---: | :---: | :---: |
| $M / M_{\text {Sun }}$ | 1.2 | 1.3 | 1.4 | 1.5 |
| $R_{S S}, \mathrm{~km}$ | 7.48 | 7.62 | 7.69 | 7.68 |
| $R_{N S}, \mathrm{~km}$ | 11.75 | 11.72 | 11.7 | 11.68 |
| $I_{S S}, \times 10^{45} \mathrm{gm} \mathrm{cm}^{2}$ | 0.65 | 0.74 | 0.825 | 0.9 |
| $I_{N S}, \times 10^{45} \mathrm{gm} \mathrm{cm}^{2}$ | 1.08 | 1.2 | 1.36 | 1.72 |

The pulsar period $P$ versus period derivative $\dot{P}$ is astrophysically measured (see, e.g. [219, 220]) distinguishes the different classes of pulsars. According to the astrophysical observations the majority of pulsars have the periods of 1 s and period derivatives of $10^{-16}$ to $10^{-14}$. Since period derivatives are in the range of about two orders
one may conclude that the neutron stars have less period derivative with compare to the strange stars.

### 4.3. Plasma magnetosphere around strange and neutron stars

The plasma magnetosphere formation around strange stars is discussed in the literature (see, for the most recent discussion, [221] ). Now assuming the presence of plasma magnetosphere around a rotating strange star we plan to study the spin-down of it due to the magnetospheric energy losses through plasma outflow along the open field lines.

Using expression from the work of [191] for the total power carried away by the relativistically moving particles one can calculate maximum value for luminosity:

$$
\begin{equation*}
\left(L_{p}\right)_{\max }=\frac{3}{2} \kappa(1-\kappa) \dot{E}_{\text {rot }}, \tag{4.15}
\end{equation*}
$$

where

$$
\begin{align*}
\dot{E}_{\text {rot }} & \equiv \frac{1}{6} \frac{\Omega^{4} B_{0}^{2} R^{6}}{c^{3} f_{R}^{2}}=\frac{1}{f_{R}^{2}}\left(\dot{E}_{\text {rot }}\right)_{\text {Newt }}, \\
f_{R} & =-\frac{3 R^{3}}{8 M^{3}}\left[\ln N_{R}^{2}+\frac{2 M}{R}\left(1+\frac{M}{R}\right)\right], \tag{4.16}
\end{align*}
$$

and $\left(\dot{E}_{\text {rot }}\right)_{\text {Newt }}$ is the standard Newtonian expression for the magneto-dipole losses in flat space-time approximation, the parameter $\kappa=2 I / R^{3}$.

Expression (4.15) could be used to investigate the rotational evolution of the star surrounded by plasma magnetosphere with predominant dipolar magnetic field
anchored in the crust which converts its rotational energy into electromagnetic emission through plasma outflow along the open field lines.

The presence of a curved spacetime has the effect of decreasing the rate of energy loss through magnetospheric plasma outflow for the strange star with comparison to that for the neutron star surrounded by plasma magnetosphere by an amount which can be easily estimated to be

$$
\begin{equation*}
\frac{\left(L_{e m}\right)_{\mathrm{sS}}}{\left(L_{e m}\right)_{\mathrm{NS}}}=\frac{[\kappa(1-\kappa)]_{\mathrm{ss}}}{[\kappa(1-\kappa)]_{\mathrm{NS}}} \frac{\left(f_{R}^{2}\right)_{N S}}{\left(f_{R}^{2}\right)_{s S}}\left(\frac{R_{\mathrm{sS}}}{R_{\mathrm{NS}}}\right)^{6} . \tag{4.17}
\end{equation*}
$$

Expression (4.17) can also be written in a more useful form in terms of the pulsar's most important observables: the period and its time derivative. In this case, in fact, using expression (4.15) and (4.13), it is not difficult to show that

$$
\begin{equation*}
(P \dot{P})_{\mathrm{ss}}=\frac{\kappa_{\mathrm{SS}}}{\kappa_{\mathrm{NS}}} \frac{1-\kappa_{\mathrm{SS}}}{1-\kappa_{\mathrm{NS}}} \frac{\left(f_{R}^{2}\right)_{\mathrm{NS}}}{\left(f_{R}^{2}\right)_{\mathrm{ss}}}\left(\frac{R_{\mathrm{SS}}}{R_{\mathrm{NS}}}\right)^{6} \frac{\widetilde{I}_{\mathrm{NS}}}{\widetilde{I}_{\mathrm{SS}}}(P \dot{P})_{\mathrm{NS}} . \tag{4.18}
\end{equation*}
$$

Also in this case it is not difficult to realize that general relativistic corrections will be introduced through the magnetic field modification and the stellar moment of inertia.

Considering slowly rotating magnetized compact star one can see that the general relativistic corrections emerging in expression (4.15) will be partly due to the magnetic field modification at the stellar surface and partly due to the compactness parameter of the strange star.

General-relativistic treatment for the structure of external and internal stellar magnetic fields including numerical results has shown that the magnetic field is
amplified by the monopolar part of gravitational field depending on the compactness of the relativistic star (see e.g. [222]). Thus for a given compact star, the effects of general relativity on electromagnetic luminosity can be characterized only by the single compactness parameter $M / R$ which is different for the neutron and strange star.

Using the data for the mass, the radius, the moment of inertia of neutron stars and strange stars from the paper [218] we have calculated the ratio of spin down of neutron star to one of the strange star on the base formula (4.18) for the compact stars of the different masses. Results are summarized in the Table 4.2 from where one can see that strange star is spinning down approximately 5 times slower that neutron star.

The pulsar period for the realistic equation of state as a function of mass is studied in [223]. The decrease of the pulsar period with the increasing of the mass of the neutron star from $1.2 M_{\text {Sun }}$ to $1.6 M_{\text {Sun }}$ is about $15 \%$ according to [223]. The changing rate of the pulsar period is much slower than the dependence of the period derivative from the compactness of the star. Steady accretion on to a magnetized and spinning star is possible only if $P>P_{b r}\left(P_{b r} \sim 50 \mathrm{~ms}\right.$ is the so-called break period for accreting the matter to the stelar surface with the typical parameters of star mass as $M=1.4 M_{\text {Sun }}$ and surface magnetic field as $B=10^{12} \mathrm{G}$ ), when the magnetohydrodynamic (e.g. RayleighTaylor and Kelvin- Helmholtz) instabilities occur at the magnetospheric boundary [224, 225, 226]. Before the onset of the instabilities, the accreting plasma can only penetrate into the magnetosphere by diffusion, with a rate much smaller than $\dot{M}$ [227]. The star should then be spun up by a steady accretion torque. The newborn strange stars resulted directly from accretion-induced collapse could rotate quite rapidly, and would spindown if the accretion rates are not very high [224]. The results of study ofthe paper [224] show that the strange stars surrounded by plasma magnetosphere rotate very rapidly due to the transfer of angular momentum of the accreting matter and consequently millisecond pulsars are rather strange stars than neutron ones. The pulsar period $P$ versus period derivative $\dot{P}$ is astrophysically measured (see, e.g. [219, 220])
distinguishes the different classes of pulsars. Since period derivatives are in the range of about two orders one may conclude that the strange stars surrounded by plasma magnetosphere have less period derivative with compare to the neutron stars.

Table 4.1.
The dependence of the ratio $(P \dot{P})_{S S} /(P \dot{P})_{N S}$ from the different parameters of the compact object: mass (in units of solar mass), radii and moment of inertia of the Strange ( $R_{S S}, I_{S S}$ ) and Neutron stars ( $R_{N S}, I_{N S}$ ). Data for strange and neutron stars are obtained from the paper [218].

| $(P \dot{P})_{S S} /(P \dot{P})_{N S}$ | 0,2053 | 0.2165 | 0.2199 | 0.2146 |
| :--- | :---: | :---: | :---: | :---: |
| $M / M_{\text {Sun }}$ | 1.2 | 1.3 | 1.4 | 1.5 |
| $R_{S S}, k m$ | 7.48 | 7.62 | 7.69 | 7.68 |
| $R_{N S}, \mathrm{~km}$ | 11.75 | 11.72 | 11.7 | 11.68 |
| $I_{S S}, \times 10^{45} \mathrm{gm} \mathrm{cm}^{2}$ | 0.65 | 0.74 | 0.825 | 0.9 |
| $I_{N S}, \times 10^{45} \mathrm{gm} \mathrm{cm}^{2}$ | 1.08 | 1.2 | 1.36 | 1.72 |

### 4.4 Conclusion

In this chapter we considered the general relativistic effects on the electromagnetic luminosity of a rotating magnetic strange star which is produced due to the rotation of the strange star with the inclined dipolar magnetic field configuration. It is shown that the effect of compactness of strange star on the electromagnetic power loss of the star is non-negligible (may have the order of tens percents of the value for the neutron star) and may help in future in distinguishing the strange star model via pulsar timing observations.

As an important application of the obtained results we have calculated energy losses of slowly rotating strange star and found that the strange star will lose more energy than typical rotating neutron star in general relativity. The obtained dependence may be combined with the astrophysical data on pulsar period slowdowns and be useful in further investigations of the possible detection/distinguish of the strange stars.

The total energy loss resulting from the magnetized star causing plasma outflow through the polar cap region, is determined through an integral over the whole polar cap area, and so it depends on both the kinetic energy density of the outflowing plasma and the surface area of the polar cap. Although general relativistic effects lead to some increase in the energy density of the outflowing plasma (due to the increase in the surface magnetic field strength for a given magnetic moment), the area of the polar cap is smaller in general relativity and the increasing the energy density of the outflowing plasma cannot compensate for the shrinking in size of the polar cap. Therefore the total energy losses of the star are significantly smaller in general relativity than in Newtonian theory. Since strange stars have bigger compactness parameter than that of neutron stars the energy loss of the strange stars is much slower.

## MAIN RESULTS AND CONCLUSIONS

According to the results of the research carried out on the theme of the doctoral dissertation "Particles and electromagnetic fields around axial-symmetric compact gravitating objects", the following conclusions are presented:

1. We have developed a new general and coordinate-independent formalism in which the shadow is described as an arbitrary polar curve expressed in terms of a Legendre expansion. It was revealed that the first five coefficients of the polynomial expansion is sufficient to describe the properties of rotating black holes shadow with the accuracy of $\sim 0.1 \%$. Our formalism does not presume any knowledge of the properties of the shadow and offers a number of routes to characterize the properties of the curve. It has been shown that the proposed definition of distortion of black holes shadow are stable under the signal noise.
2. The analytical expressions for the vacuum electromagnetic fields of deformed rotating black holes in the external asymptotically uniform magnetic field hads been obtained. It has been revealed that the induced electric field around the deformed black hole depends on deformation parameter linearly, and magnetic field squared.
3. An upper limit for the deformation parameter for the rotating non-Kerr black hole has been obtained through comparison of the observable values of the radius of innermost stable circular orbits with the theoretical results obtained in the dissertation as $\varepsilon \leq 22$.
4. It has been obtained the silhouettes of the rotating black holes shadow in the presence of an inhomogeneous plasma, which can be used to identify additional asymmetries in the shape of the shadow and retrieve information on the plasma parameters and the central compact object.
5. Expressions for energy and momentum, as well as radii of innermost stable circular orbits of charged particles in the vicinity of a black hole with gravitomagnetic charge immersed in external magnetic field has been obtained. It has been established that due to the existence of gravitomagnetic charge particle are prevented from acceleration to infinitely high energies.
6. It has been shown that In the presence of a plasma the observed shape and size of the shadow changes depending on (i) plasma parameters, (ii) black hole spin, and (iii) inclination angle between the observer plane and the axis of rotation of the black hole. It has been found that the observed size of the shadow of the black hole decreases due to the refraction of electromagnetic radiation in a plasma environment. It was shown that with the increase of the dimensionless plasma parameter, the maximum value of the energy emission rate from the black hole decreases due to the decrease of the size of the black hole shadow.
7. It was shown that for the high efficiency of the ultrahigh-energy processes relative to distant observers, both the non-existence of the horizon, and the strong rotational effects are necessary; it was also shown that significant magnification of the efficiency of the ultra-high energy collisions is possible due to additional electromagnetic phenomena influencing collisions of charged particles.
8. It was shown that energy extraction through Penrose process is more realistic process among the energy extraction mechanisms from the rotating black hole in Horava scenario; moreover, due to the Horava gravity correction particles could be prevented from the infinite acceleration.
9. It was shown that the effect of compactness of strange star on the electromagnetic power loss of the star is non-negligible and may help in future in distinguishing the strange star model via pulsar timing observations. It was found that the relativistic strange star would lose more energy than typical rotating neutron star in general relativity. The obtained dependence may be useful in further investigations of the possible detection/distinguishment of the strange stars.

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Axial symmetric black hole immersed in uniform magnetic field

## Electromagnetic field around rotating non-Kerr black hole in external

 uniform magnetic field. Here we will exploit the existence in this spacetime of a timelike Killing vector $\xi_{(t)}^{\alpha}$ and spacelike one $\xi_{(\varphi)}^{\alpha}$ being responsible for stationarity and axial symmetry of geometry, such that they satisfy the Killing equations$$
\begin{equation*}
\xi_{\alpha ; \beta}+\xi_{\beta ; \alpha}=0, \tag{A.1.1}
\end{equation*}
$$

which according toWald method [180] gives a right to write the solution of vacuum Maxwell equations $\square A^{\mu}=0$ for the vector potential $A_{\mu}$ of the electromagnetic field in the Lorentz gauge in the simple form

$$
\begin{equation*}
A^{\alpha}=C_{1} \xi_{(t)}^{\alpha}+C_{2} \xi_{(\varphi)}^{\alpha} . \tag{A.1.2}
\end{equation*}
$$

The constant $C_{2}=B / 2$, where gravitational source is immersed in the uniform magnetic field B being parallel to its axis of rotation. The value of the remaining constant $C_{1}$ can be easily calculated from the asymptotic properties of spacetime (1.88) at the infinity. Indeed in order to find the remaining constant one can use the electrical neutrality of the black hole $4 \pi Q=0$ evaluating the value of the integral through the spherical surface at the asymptotic infinity. Then one can easily get the value of constant $C_{2}=a B$.

Thus the 4 -vector potential $A_{\mu}$ of the electromagnetic field will take the following

Form

$$
\begin{align*}
& A_{0}=-a B \frac{\Sigma^{2}-2 M r+M r \sin ^{2} \theta}{\Sigma^{2}}(1+h), A_{1}=A_{2}=0, \\
& A_{3}=\frac{1}{2} B \sin ^{2} \theta\left[\Sigma^{2}+\frac{\left(2 M r+\Sigma^{2}\right) \sin ^{2} \theta-4 M r}{\Sigma^{2}} a^{2}(1+h)\right] . \tag{A.1.3}
\end{align*}
$$

The orthonormal components of the electromagnetic fields measured by zero angular momentum observers (ZAMO) with four velocity components

$$
\begin{align*}
\left(u^{\alpha}\right)_{\text {ZAMO }} \equiv & \left(\sqrt{\frac{\mathcal{R}}{\Sigma^{2}(1+h)\left(\Delta+a^{2} h \sin ^{2} \theta\right)}}, 0,0,\right. \\
& \left.-\frac{2 M a r \sqrt{1+h}}{\sqrt{\Sigma^{2}\left(\Delta+a^{2} h \sin ^{2} \theta\right) \mathcal{R}}}\right),  \tag{A.1.4}\\
\left(u_{\alpha}\right)_{\text {ZAMO }} \equiv & \left(\sqrt{\frac{\Sigma^{2}(1+h)\left(\Delta+a^{2} h \sin ^{2} \theta\right)}{\mathcal{R}}}, 0,0,0\right) \tag{A.1.5}
\end{align*}
$$

are given by expressions

$$
\begin{align*}
& E^{\hat{r}}=\frac{a B M}{256 \sqrt{\mathcal{R}} \Sigma^{6}(1+h)} \mathcal{P}_{1},  \tag{A.1.6}\\
& E^{\hat{\theta}}=\frac{a B M r}{\Sigma^{2} \sqrt{\left(\Delta+a^{2} \sin ^{2} \theta\right) \mathcal{R}}} \mathcal{P}_{2},  \tag{A.1.7}\\
& B^{\hat{r}}=\frac{B \cos \theta}{\Sigma^{4} \sqrt{\mathcal{R}}} \mathcal{P}_{3},  \tag{A.1.8}\\
& B^{\hat{\theta}}=\frac{B \sin \theta}{2 \Sigma^{8}} \sqrt{\frac{\Delta+a^{2} h \sin ^{2} \theta}{\mathcal{R}}} \mathcal{P}_{4}, \tag{A.1.9}
\end{align*}
$$

where the following notations

$$
\mathcal{R}=\Sigma^{4}+a^{2}(1+h)\left(2 M r+\Sigma^{2}\right) \sin ^{2} \theta,
$$

have been introduced. The expressions for the quantities $P_{1}, P_{2}, P_{3}, P_{4}$ can be found in Ref. [7]. The electromagnetic field (A.1.6)-(A.1.9) depends on angular momentum and deformation parameter in complex way. Astrophysically it is interesting to know the limiting cases of expressions (A.1.6)-(A.1.9), for example in either linear or quadratic approximation $O\left(a^{2} / r^{2} ; \varepsilon\right)$ in order to give physical interpretation of possible physical processes near the rotating non-Kerr compact object, where they take following form:

$$
\begin{align*}
& E^{\hat{r}} \rightarrow \frac{a B M}{r^{3}}\left[\cos ^{2} \theta-3+3\left(\frac{M^{2}}{r^{2}}-\frac{2 M^{3}}{r^{3}}+\frac{M^{3}}{r^{3}} \sin ^{2} \theta\right) \epsilon\right]  \tag{A.1.10}\\
& E^{\hat{\theta}} \rightarrow \frac{a B M}{r} \frac{2 r^{3}+\epsilon M^{3}}{r^{3} \sqrt{r^{2}-2 M r}} \sin 2 \theta  \tag{A.1.11}\\
& B^{\hat{r}} \rightarrow B \cos \theta\left[1+\frac{a^{2}}{r^{2}}\left(1-\frac{2 M}{r}\right)\left(1+\frac{\epsilon M^{3}}{r^{3}}\right)-\frac{2 a^{2} M}{r^{3}} \cos 2 \theta\right]  \tag{A.1.12}\\
& B^{\hat{\theta}} \rightarrow B \sin \theta\left[1+\frac{2 a^{2} M}{r^{3}}\left(1+\frac{4 \epsilon M^{3}}{r^{3}}\right)-\frac{2 a^{2} M}{2 r^{3}} \sin ^{2} \theta\right] \tag{A.1.13}
\end{align*}
$$

In the limit of flat spacetime, i.e. for $M / r \rightarrow 0$, expressions (A.1.6)-(A.1.9) give

$$
\begin{equation*}
E^{\hat{r}}=E^{\hat{\theta}} \rightarrow 0, B^{\hat{f}} \rightarrow B \cos \theta, \quad B^{\hat{\theta}} \rightarrow B \sin \theta \tag{A.1.14}
\end{equation*}
$$

As expected, expressions (A.1.14) coincide with the solutions for the homogeneous magnetic field in the Newtonian spacetime.

Black hole with gravitomagnetic charge in uniform magnetic field. Consider the electromagnetic field around black hole with non vanishing gravitomagnetic charge. In this case one can write the solution of vacuum Maxwell equations $\square A^{\mu}=0$ for the vector potential $A_{\mu}$ of the electromagnetic field in the Lorentz gauge in the simple form

$$
\begin{equation*}
A^{\alpha}=C_{1} \xi_{(t)}^{\alpha}+C_{2} \xi_{(\varphi)}^{\alpha} . \tag{A.1.15}
\end{equation*}
$$

The constant $C_{2}=B / 2$. The value of the remaining constant $C_{1}$ can be easily calculated from the asymptotic properties of spacetime (1.98) at the infinity. Indeed in order to find the remaining constant one can use the electrical neutrality of the black hole

$$
\begin{equation*}
4 \pi Q=C_{1} \oint \Gamma_{\beta \gamma}^{\alpha} \tau_{\alpha} m^{\beta} \xi_{(t)}^{\gamma}(\tau k) d S+\frac{B}{2} \oint \Gamma_{\beta \gamma}^{\alpha} \tau_{\alpha} m^{\beta} \xi_{(\phi)}^{\gamma}(\tau k) d S=0 \tag{A.1.16}
\end{equation*}
$$

evaluating the value of the integral through the spherical surface at the asymptotic infinity. Here the equality $\xi_{\beta ; \alpha}=-\xi_{\alpha ; \beta}=-\Gamma_{\alpha \beta}^{\gamma} \xi_{\gamma}$ following from the Killing equation was used, and element of an arbitrary 2 -surface $d S^{\alpha \beta}$ is represented in the form

$$
\begin{equation*}
d S^{\alpha \beta}=-\tau^{\alpha} \wedge m^{\beta}(\tau k) d S+\eta^{\alpha \beta \mu \nu} \tau_{\mu} n_{\nu} \sqrt{1+(\tau k)^{2}} d S, \tag{A.1.17}
\end{equation*}
$$

and the following couples

$$
\begin{aligned}
m_{\alpha} & =\frac{\eta_{\lambda \alpha \mu \nu} \tau^{\lambda} n^{\mu} k^{\nu}}{\sqrt{1+(\tau k)^{2}}} \\
n_{\alpha} & =\frac{\eta_{\lambda \alpha \mu \nu} \tau^{\lambda} k^{\mu} m^{\nu}}{\sqrt{1+(\tau k)^{2}}} \\
k^{\alpha} & =-(\tau k) \tau^{\alpha}+\sqrt{1+(\tau k)^{2}} \eta^{\mu \alpha \rho \nu} \tau_{\mu} m_{\rho} n_{\nu}
\end{aligned}
$$

are established between the triple $\{\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{n}\}$ of vectors, $n^{\alpha}$ is normal to 2 -surface, spacelike vector $m^{\alpha}$ belongs to the given 2-surface and is orthogonal to the four velocity of observer $\tau^{\alpha}$, a unit spacelike four-vector $k^{\alpha}$ belongs to the surface and is orthogonal to $m^{\alpha}, d S$ is invariant element of surface, $\wedge$ denotes the wedge product, $*$ is for the dual element, $\eta_{\alpha \beta \gamma \delta}$ is the pseudo-tensorial expression for the Levi-Civita symbol $\varepsilon_{\alpha \beta \gamma \delta}$. Using the asymptotic values of the vectors, one can get the value of constant $C_{1}=0$. Thus the 4 -vector potential $A_{\alpha}$ of the electromagnetic field will take the following form

$$
\begin{align*}
& A_{0}=-\frac{\Delta}{\Sigma} B l \cos \theta  \tag{A.1.18}\\
& A_{3}=\frac{1}{2} \Sigma^{2} \sin ^{2} \theta-2 \Delta l^{2} \cos ^{2} \theta \tag{A.1.19}
\end{align*}
$$

The orthonormal components of the electromagnetic fields measured by zero angular momentum observers (ZAMO) with four velocity components

$$
\begin{align*}
& \left(\tau^{\alpha}\right)_{\text {ZAMO }} \equiv\left(\sqrt{\frac{\mathcal{R}}{\Delta \Sigma \sin ^{2} \theta}}, 0,0, \frac{2 \Delta l \cos \theta}{\sqrt{\Delta \Sigma \mathcal{R} \sin ^{2} \theta}}\right)  \tag{A.1.20}\\
& \left(\tau_{\alpha}\right)_{\text {ZAMO }} \equiv\left(\sqrt{\frac{\Delta \Sigma \sin ^{2} \theta}{\mathcal{R}}}, 0,0,0\right) \tag{A.1.21}
\end{align*}
$$

are given by expressions

$$
\begin{align*}
E^{\hat{r}} & =-\frac{B r l}{\sqrt{\mathcal{R}}}\left(1-\frac{M}{r}\right) \sin 2 \theta  \tag{A.1.22}\\
E^{\hat{\theta}} & =\frac{B l}{\Sigma^{2}} \sqrt{\frac{\Delta}{\mathcal{R}}}\left[\Sigma^{2}+\left(\Sigma^{2}-2 \Delta l^{2} \cos \theta\right) \frac{2 \cos \theta}{\sin ^{2} \theta}\right] \sin ^{2} \theta  \tag{A.1.23}\\
B^{\hat{r}} & =\frac{B \tan \theta}{\Sigma \sqrt{\mathcal{R}}}\left(\mathcal{R}-\Sigma^{2}\right)  \tag{A.1.24}\\
B^{\hat{\theta}} & =\frac{B r}{\Sigma^{2}} \sqrt{\frac{\Delta}{\mathcal{R}}} \cos ^{2} \theta\left\{\left[\Delta-\Sigma\left(1-\frac{M}{r}\right)\right] 4 l^{2}+\Sigma^{2} \tan ^{2} \theta\right\} \tag{A.1.25}
\end{align*}
$$

where the following notation has been used:

$$
\mathcal{R}=\Sigma^{2} \sin ^{2} \theta-4 \Delta l^{2} \cos ^{2} \theta
$$

Astrophysically it is interesting to know the limiting cases of expressions (A.1.22)-(A.1.25), for example in either linear or quadratic approximation in order to give physical interpretation of possible physical processes near the slowly rotating relativistic compact objects, where they take the following form:

$$
\begin{align*}
& E^{\hat{r}}=\frac{2 B l \cos \theta}{r}\left(1-\frac{3 M}{r}\right)  \tag{A.1.26}\\
& E^{\hat{\theta}}=\frac{B l \sin \theta}{r}\left(1+\frac{2 \cos \theta}{\sin ^{2} \theta}\right)  \tag{A.1.27}\\
& B^{\hat{r}}=B \cos \theta\left[1+2 \frac{l^{2}}{r^{2}} \frac{1}{\sin ^{2} \theta}\right] \tag{A.1.28}
\end{align*}
$$

$$
\begin{equation*}
B^{\hat{\theta}}=B \sin \theta\left[1-\frac{M}{r}+\frac{1}{16 r^{2} \sin ^{2} \theta}\left(4 l^{2}-4 M^{2}+4\left(7 l^{2}+M^{2}\right) \cos 2 \theta\right) A\right] \tag{A.1.29}
\end{equation*}
$$

In the limit of flat spacetime expressions (A.1.26)-(A.1.29) give

$$
\begin{gather*}
E^{\hat{r}}=E^{\hat{\theta}} \rightarrow 0,  \tag{A.1.30}\\
B^{\hat{r}} \rightarrow B \cos \theta, \quad B^{\hat{\theta}} \rightarrow B \sin \theta . \tag{A.1.31}
\end{gather*}
$$

As expected, expressions (A.1.30)-(A.1.31) coincide with the solutions for the homogeneous magnetic field in Newtonian spacetime.

The dynamical equation for a charged particle motion can be written as

$$
\begin{equation*}
m \frac{d u^{\mu}}{d \tau}=q F_{\nu}^{\mu} u^{\nu} \tag{A.1.32}
\end{equation*}
$$

where $\tau$ is the proper time, $u^{\mu}$ is the 4 -velocity of a charged particle, $u^{\mu} u_{\mu}=-1, q$ and $m$ are its charge and mass, respectively. $F_{\alpha \beta}=A_{\beta, \alpha}-A_{\alpha, \beta}=$ is the antisymmetric tensor of the electromagnetic field, which has the following four independent components

$$
\begin{aligned}
& F_{01}=\frac{B}{\Sigma^{2}} 2 l[(r-M)-\Delta r] \cos \theta, \\
& F_{02}=\frac{B}{\Sigma} \Delta l \sin \theta, \\
& F_{13}=-\frac{B}{\Sigma^{2}} 8 l^{2}[(r-M) \Sigma-\Delta r] \cos ^{2} \theta+B r \sin ^{2} \theta, \\
& F_{23}=\frac{B}{\Sigma}\left(4 l^{2} \Delta+\frac{1}{2} \Sigma^{2}\right) \sin 2 \theta .
\end{aligned}
$$

Uniform magnetic field in the KS naked singularity spacetime. First, we have to give the form of the electromagnetic field related to the asymptotically uniform magnetic field. The situation is complicated by the fact that the KS spacetimes are not Ricci flat [132]. The Ricci tensor of the KS spacetimes has the following nonzero components:

$$
\begin{align*}
& R_{t}^{t}=R_{r}^{r}=\frac{3\left(2+6 r^{3} \omega+r^{6} \omega^{2}-r^{6} \omega^{2} A^{3}\right)}{r^{6} \omega A^{3}}  \tag{A.1.33}\\
& R_{\theta}^{\theta}=R_{\phi}^{\phi}=\frac{3\left(2+r^{3} \omega-r^{3} \omega A\right)}{r^{3} A} \tag{A.1.34}
\end{align*}
$$

where $A(r, \omega)$ is given by Eq. (A.2.15).
Let us consider a KS naked singularity or black hole in an external asymptotically uniform magnetic field. In order to find the related vector potential of the electromagnetic field we apply the Wald method that uses the Killing vectors of the spacetime [180]. The Killing vector field $\xi^{\mu}$, being an infinitesimal generator of an isometry, fulfills the equation

$$
\begin{equation*}
\xi_{\alpha ; \beta}+\xi_{\beta ; \alpha}=0 . \tag{A.1.35}
\end{equation*}
$$

The equation (A.1.35) implies the equation

$$
\begin{equation*}
\xi_{\alpha ; \beta ; \gamma}-\xi_{\alpha ; \gamma ; \beta}=-\xi^{\lambda} R_{\lambda \alpha \beta \gamma} \tag{A.1.36}
\end{equation*}
$$

containing the Riemann curvature tensor $R_{\alpha \beta \gamma \delta}$. Therefore, the Killing vectors reflecting the symmetries of the KS spacetimes have to satisfy the relation

$$
\begin{equation*}
\xi_{; \beta}^{\alpha ; \beta}=R_{\mu}^{\alpha} \xi^{\mu} . \tag{A.1.37}
\end{equation*}
$$

The electromagnetic potential satisfying the Lorentz calibration condition has to fulfill the relation [180]

$$
\begin{equation*}
A_{; \beta}^{\alpha ; \beta}=-R_{\mu}^{\alpha} A^{\mu} . \tag{A.1.38}
\end{equation*}
$$

Asymptotically (for $\rightarrow \infty$ ) these two equations reduce to the homogeneous form [1,147]

$$
\begin{equation*}
\xi_{; \beta}^{\alpha ; \beta}=A_{; \beta}^{\alpha ; \beta}=0 . \tag{A.1.39}
\end{equation*}
$$

In order to find the solution of the Maxwell equations for the vector potential $A_{\alpha}$, one may use the following anzatz:

$$
\begin{equation*}
A^{\alpha}=C_{1} \xi_{(t)}^{\alpha}+C_{2} \xi_{(\phi)}^{\alpha}-a^{\alpha} \tag{A.1.40}
\end{equation*}
$$

where $a_{\alpha}$ is the correction due to the non-zero Ricci tensor of the KS space-time. It can be found from the equations

$$
\begin{equation*}
a^{\alpha ; \beta}{ }_{; \beta}=C_{1} R_{\gamma}^{\alpha} \xi_{(t)}^{\gamma} \tag{A.1.41}
\end{equation*}
$$

$$
\begin{equation*}
a_{; \beta}^{\alpha ; \beta}=C_{2} R_{\gamma}^{\alpha} \xi_{(\phi)}^{\gamma} \tag{A.1.42}
\end{equation*}
$$

The first two terms in this equation are related to the homogeneous solution of the equation that is directly related to the Killing vectors of the KS spacetimes. The third term is the partial solution related to the Ricci tensor of the spacetime; in our case it can be found by numerical calculations in fully general case, but we can find analytic approximative solutions. The constants related to the Killing vectors can be found easily - the constant $C_{2}=B / 2$, if we have the gravitational source immersed in the uniform magnetic field $B$ that is parallel to the axis of rotation corresponding to the axial symmetry of the spacetime. The remaining constant $C_{1}$ has to vanish as can be easily shown due to the asymptotic properties of the KS spacetimes at the infinity.

Solving the equation (A.1.41)-(A.1.42) by using the expressions (A.1.33)(A.1.34), one can get the full expression for the electromagnetic potential. Here we give the approximate formulae that can be presented in analytic form.

Considering the first order approximation in $\omega^{-1}$ of the KS-geometry Ricci tensor contribution, corresponding to the KS black holes close to the Schwarzschild limit, the vector potential of the electromagnetic field takes the form [148]

$$
\begin{equation*}
A^{\alpha}=\frac{B}{2}\left(0,0,0,1+\frac{3 M^{2}}{10 \omega r^{4}}\right) . \tag{A.1.43}
\end{equation*}
$$

There is only one non-zero covariant component of the vector potential

$$
\begin{equation*}
A_{\phi}=\frac{B}{2} r^{2} \sin ^{2} \theta\left(1+\frac{3 M^{2}}{10 \omega r^{4}}\right) . \tag{A.1.44}
\end{equation*}
$$

The solution of the equation (A.1.42) then gets the following form

$$
\begin{equation*}
A^{\alpha}=\frac{B}{2}\left(0,0,0,1+3 \sqrt{\frac{\omega}{r^{3}}}+\frac{\omega}{r^{2}}+\mathcal{O}\left(\omega^{2}\right)\right) . \tag{A.1.45}
\end{equation*}
$$

For the situations we consider in this workthe contribution of the vector $a_{\alpha}$ to the total electromagnetic field around the KS naked singularity, as well as to the particle motion and collisional processes is less then $5 \%$ [11].

Therefore, it is enough to consider here the basic approximation where the vector potential is represented in the simple form proportional to the axial Killing vector. Such an approximation is precise asymptotically at large distances from the KS naked singularities, and its precision is high enough in arbitrary positions. In this approximation we thus investigate the role of the vector potential

$$
\begin{equation*}
A_{\phi}=\frac{B}{2} r^{2} \sin ^{2} \theta . \tag{A.1.46}
\end{equation*}
$$


[^0]:    ${ }^{1}$ Review of international scientific researches on dissertation subject composed on the basis of the following sources: http://arxiv.org; https://webofknowledge.com; https://scholar/google/com. J. Physical Review Letters; J. Physical Review D; J. Monthly Notices of Royal Astronomical Society; J. Astrophysical Journal; Ж. Astrophysics and Space Science; J. International Joournal of Modern Physics D; and others.

[^1]:    ${ }^{2}$ Strictly speaking, also a horizonless object such as a gravastar [19] would lead to a shadow. However, rather exotic assumptions on the heat capacity of the gravastar's surface are needed to justify a lack of emission from such a surface [20]; gravitational-wave emission would unambigously signal the presence of an event horizon [21].

[^2]:    ${ }^{3}$ http://eventhorizontelescope.org/
    4 http://blackholecam.org/

[^3]:    ${ }^{5}$ The accuracy of the choice of the potential of the electromagnetic field around KS naked singularity is shown in Appendix 1.

